Homework 4

Due: Wednesday, October 7

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. Prove that

$$G = \{a + \sqrt{2}b \mid a, b \in \mathbb{Q},\ a \text{ and } b \text{ not both zero}\}$$

is a subgroup of $(\mathbb{R}\backslash\{0\}, \times)$.

Recall that $\mathbb{Q}$ denotes the set of rational numbers, while $\mathbb{R}$ denotes the set of real numbers. You can use without a proof that $\sqrt{2}$ is irrational.

Problem 2. Prove that a subgroup $H \subset \mathbb{Z}$ (with respect to addition) such that

$$6, 10, 15 \in H$$

must coincide with $\mathbb{Z}$.

Problem 3. *a)* Compute the Cayley table of the group $(\mathbb{Z}_4, +)$.

b) Compute the Cayley table of the group $(\mathbb{Z}_5^\times, \times)$.

Problem 4. Prove that the group $(\mathbb{Z}_8^\times, \times)$ is not cyclic.

Problem 5. *a)* Prove that the congruence class $2 \in \mathbb{Z}_{13}$ generates (multiplicatively) the group $(\mathbb{Z}_{13}^\times, \times)$.

b) Find all other generators of this group.