Homework 5

Due: Wednesday, October 14

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. List all the elements of the following cyclic subgroups:

a) $\langle 10\rangle \subset (\mathbb{Z}_{15}, +)$

b) $\langle 7\rangle \subset (\mathbb{Z}_9, +)$

Problem 2. H is a cyclic subgroup of $(\mathbb{Z}_{16}^{\times}, \times)$. What are the possible sizes $|H|$ of H?

Problem 3. *a*) Prove that the product group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, of $(\mathbb{Z}_2, +)$ and $(\mathbb{Z}_2, +)$ is not cyclic.

b) Consider the product group $G = \mathbb{Z}_2 \times \mathbb{Z}_3$ of the corresponding **additive** groups. Prove that G is cyclic with the generator $(1,1) \in \mathbb{Z}_2 \times \mathbb{Z}_3$.

Problem 4. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

be elements of $GL_2(\mathbb{R})$ (considered as usual with respect to the matrix multiplication). Show that A and B have finite order, but their product AB has infinite order.

Problem 5. Let $(G, *)$ be a **commutative** group, and $x, y \in G$ are two elements of a finite order. Prove that the element $x * y \in G$ also has a finite order.