Lecture 15

Reference: Judson, Chapter 5

Permutation group

Parity of permutation

On of the key characteristics of a permutation $\sigma \in S_n$ is its parity — it turns out that we can split all permutation into odd and even, and this splitting satisfies very nice properties.

Let $\sigma \in S_n$ be any permutation. Draw its mapping diagram. We allow arrows to move right/left as long as they always point into the bottom half-plane (below you can see 6 diagrams for S_3 going from top to bottom).

Proposition 1: Parity of intersection count

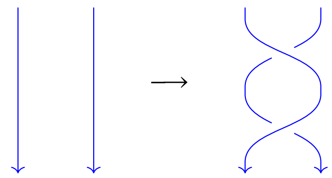
Given any $\sigma \in S_n$ consider its mapping diagram. Let I be the number of intersection points of arrows[a]. Then for any other drawing of the mapping diagram with J intersection points, we will have

$$I = J \mod 2.$$

i.e. I and J have the same parity

[a] To be pedantic: we assume that arrows are smooth curves, and they are not tangent at the intersection points.

Proof. The reason why parity of I does not change, when we *move arrow around* is best explained in the following picture

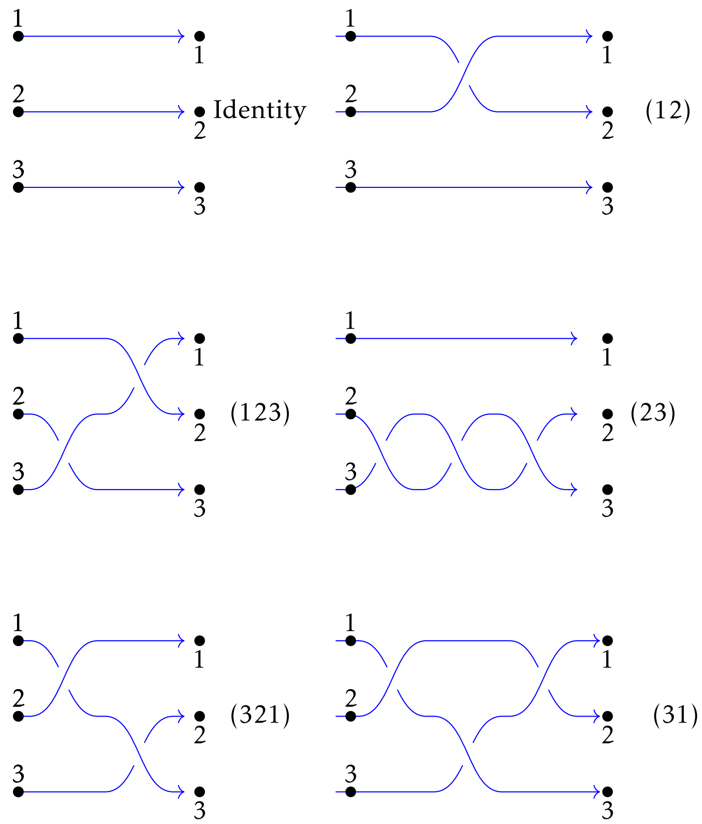


This picture shows that intersection points appear and disappear in **pairs**, thus the parity of the number of the intersection points does not change. □

Definition 1: Parity & sign of permutation

Consider **any** mapping diagram of a permutation $\sigma \in S_n$. Let I be the number of intersection points of the arrows. The **parity** of permutation σ is the parity of I.

Equivalently, we can express the parity of permutation in terms of its **sign** $\operatorname{sgn}(\sigma) \in \{-1, +1\}$. Namely, if σ is even, we will say that $\operatorname{sgn}(\sigma) = 1$, and if σ is odd, we will say $\operatorname{sgn}(\sigma) = -1$.



Example 1

For the above diagrams of permutations in S_3 we have numbers of intersections

0	1
2	3
2	3

Thus the permutations in the left column are **even** and permutations in the right column are **odd**.

NB 1: Proposition 1 ensures that definition of parity is **correct** — and the result does not depend on the presentation of the mapping diagram.

Example 2

1. The identity permutation which fixes all elements $id \in S_n$ has a diagram without any intersection points. Thus id is even.
2. If $\tau = (a,b)$ is a transposition (cycle of length 2) swapping elements a and b, then τ is odd. Indeed: Assume $a < b$ for concreteness. Then, there is a mapping diagram of τ such that the arrow $b \to a$ intersects every arrow $c \to c$ for $a < c < b$ exactly once, and also arrow $a \to b$ intersects every such arrow $c \to c$ exactly once. This gives $2(b-a-1)$ intersection points. Plus there is a unique intersection point of arrows $a \to b$ and $b \to a$ (see example $(3,1)$ above). Overall this gives an odd number of intersections.

The following proposition is the key to understanding signs of permutations.

Proposition 2

Given any permutations $\sigma, \tau \in S_n$ we have

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)$$

Proof. If mapping diagram for τ has $I(\tau)$ intersection points, and mapping diagram for σ has $I(\sigma)$ intersection points, then the mapping diagram for $\sigma\tau$ obtained from "stacking" diagrams for σ and τ together, will have $I(\tau)+I(\sigma)$ intersection points. This implies the desired identity. □

Remark 1

lternatively, you can think about the above claim as follows:

product of even permutations is even

product of odd permutations is even

product of an odd and an even permutations (in any order) is odd

Example 3

ycle $\mu = (a_1, \dots, a_k)$ of length k can be factored into a product of $(k-1)$ transpositions (Problem #2 from Homework #7). Let us call these transpositions τ_i. Therefore

$$\operatorname{sgn}(\mu) = \operatorname{sgn}(\tau_1)\dots\operatorname{sgn}(\tau_{k-1}) = (-1)^{k-1}.$$

Thus μ is even if k is odd, and vice versa μ is odd if k is even.

Problem 1: Let us think of S_4 as the group of symmetries of a tetrahedron. Then each rigid motion of a tetrahedron can be classified as odd or even, according to the parity of the corresponding permutation. Give a geometric characterization distinguishing the odd rigid motions from the even.

Example 4

Consider permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 6 & 2 & 5 \end{pmatrix}.$$

Of course we could draw its diagram and try to find its parity by intersection points count. But a much more efficient approach relies on Proposition 2 and the factorization of σ into independent cycles:

First we factorize σ into cycles

$$\sigma = (1,3)(2,4,6,5)$$

Both cycles $(1,3)$ and $(2,4,6,5)$ are of even length, thus they have odd parity by the previous example. Therefore σ as a product of two odd permutations is **even** itself.

Problem 2: Let $A_n \subset S_n$ be the subset consisting of all even permutations. Prove that A_n is a subgroup.