Lecture 18

Reference: Judson, Chapter 11

Cosets of Kernels

In this section we will relate two notions in group theory — homomorphisms and kernels — through the partition of a group into cosets of the kernel.

Let

$$f : G \to H$$

be a *homomorphism* of groups (recall that it means that for any $x, y \in G$, we have $f(xy) = f(x)f(y)$. Consider also the *kernel* of f:

$$\mathrm{Ker}(f) = \{x \in G \mid f(x) = e_H\}.$$

Since $\mathrm{Ker}(f) \subset G$ is a subgroup, we can consider the partition of G into (left) cosets, which are, by definition, subsets

$$g\,\mathrm{Ker}(f) = \{gx \mid x \in \mathrm{Ker}(f)\}.$$

Question 1

What are the left cosets of $\mathrm{Ker}(f)$?

Proposition 1

Elements $x, y \in G$ belong to the same left coset of $\mathrm{Ker}(f)$ if and only if

$$\boxed{f(x) = f(y)}$$

Proof. Elements x and y belong to the same left coset of $\mathrm{Ker}(f)$ *if and only if* one can multiply x **from the left** by an element k in $\mathrm{Ker}(f)$ and obtain y:

$$xk = y.$$

This happens *if and only if* $y^{-1}x \in \mathrm{Ker}(f)$.
By definition, the latter holds if and only if

$$f(y{-}1x) = e_H.$$

Using the defining property of homomorphisms, we find that it happens if and only if

$$f(y)^{-1}f(x) = e_H \Longleftrightarrow f(x) = f(y).$$

Since at each step of our proof we used **if and only if** statements, we have proved the proposition in both directions. □

Corollary 1

Left cosets of $\mathrm{Ker}(f) \subset G$ are in 1-to-1 correspondence with the image $\mathfrak{Im}(f) \subset H$.

Remark 1

The entire discussion above holds verbatim for the *right* cosets.

Definition 1: L

t $f : X \to Y$ be a map between **sets**. A **fiber** over an element $y \in Y$ is the set of all elements in X mapped to this y:

$$f^{-1}(y) = \{x \in X \mid f(x) = y\} \subset X.$$

Proposition 1 can be now stated as follows: left cosets of $\mathrm{Ker}(f)$ are precisely the nonempty fibers of f.

Example 1

Consider the *sign* homomorphism:

$$\mathrm{sgn}\colon S_n \to \{+1, -1\}.$$

Its kernel is the alternating group A_n. This group has two cosets:

$$A_n = \{\text{even permutations}\}, \quad \{\text{odd permutation}\}$$

which correspond respectively to elements $+1$ and -1 of group $\{+1, -1\}$.

Example 2

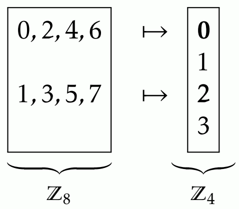
Consider a homomorphism

$$f\colon \mathbb{Z}_8 \to \mathbb{Z}_4$$

defined by

$$f(x) = 2x \mod 4$$

Elementwise this map can be presented as follows:



Thus we see that both fibers $f^{-1}(0)$ and $f^{-1}(2)$ have 4 elements (the size of $\mathrm{Ker}(f)$), while the fibers $f^{-1}(1)$ and $f^{-1}(3)$ are empty, because 1 and 3 are not in the image of f.

There is an important enumerative corollary of the above proposition.

Proposition 2

If $f\colon G \to H$ is a homomorphism, then

- $[G : \mathrm{Ker}(f)] = |\mathrm{Im}(f)|$
- $|G| = |\mathrm{Im}(f)| \cdot |\mathrm{Ker}(f)|$

Proof. From Proposition 1 we know that the left cosets of $\mathrm{Ker}(f)$ in 1-to-1 correspondence with $|Im(f)|$. This proofs the first claim.
To prove the second claim, we invoke Lagrange's theorem, which states that for any subgroup $K \subset G$

$$|G| = [G : K] \cdot |K|,$$

apply it to $K = \mathrm{Ker}(f)$, and substitute $[G : \mathrm{Ker}(f)]$ from the first part. □

This proposition can be useful in studying homomorphisms between different groups.

Example 3

Consider a homomorphism $f\colon S_5 \to \mathbb{Z}_7$. On one hand, by Lagrange's theorem the size of subgroup $\mathfrak{Im}(f) \subset \mathbb{Z}_7$ divides 7.
On the other hand, by the proposition

$$\underbrace{|S_5|}_{5!=120} = |\mathrm{Ker}(f)||\mathfrak{Im}(f)|,$$

thus $|\mathfrak{Im}(f)|$ is a factor of 120.
The only option this leaves for $|\mathfrak{Im}(f)|$ is 1, hence f is a *trivial* homomorphism, i.e., it maps every element of S_5 to the identity $0 \in \mathbb{Z}_7$.