Lecture 19

Reference: Judson, Chapter 10

Normal subgroups

Let $N \subset G$ be a subgroup. Recall a definition.

Definition 1

A subgroup $N \subset G$ is *normal* if for any element $x \in N$ and any $g \in G$ the conjugate

$$gxg^{-1}$$

also belongs to N.

Example 1

1. If group G is abelian, then any its subgroup is automatically normal, since $gxg^{-1} = x$.
2. The alternating group $A_n \subset S_n$ is normal, since the conjugates

$$\text{odd} \cdot \text{even} \cdot \text{odd}^{-1} \qquad \text{even} \cdot \text{even} \cdot \text{even}^{-1}$$

   are always even.
3. If N is the kernel of a homomorphism $f : G \to H$, then N is normal. Indeed, if $f(x) = e_H$, then

$$f(gxg^{-1}) = f(g)f(g^{-1}) = e_H,$$

   so $gxg^{-1} \in \mathrm{Ker}(f)$.
4. The subgroup $SL_n(\mathbb{R})$ of $(GL_n(\mathbb{R}), \cdot)$ is defined as the kernel of the homomorphism $\det : (GL_n(\mathbb{R}), \cdot) \to (\mathbb{R}^\times, \cdot)$,, therefore it is normal.

The following proposition gives several equivalent convenient ways to state that $N \subset G$ is normal

Proposition 1

Denote

$$gNg^{-1} := \{gxg^{-1} \mid x \in N\} \subset G.$$

The following properties of a subgroup $N \subset G$ are equivalent

1. N is normal;
2. For any $g \in G$ we have

$$gNg^{-1} = N$$

3. For any $g \in G$ the left and right cosets of N generated by g coincide:

$$gN = Ng.$$

Proof. We start by proving the implication $1 \Rightarrow 2$. If N is normal, then we have that for every $g \in G$ $gNg^{-1} \subset N$. It remains to prove the reverse inclusion. Let $x \in N$, and consider the element $k = g^{-1}xg$. Since N is normal, we have $k \in N$. Then $x = gkg^{-1}$ is an element of gNg^{-1}, so $N \subset gNg^{-1}$.
We now prove $2 \Rightarrow 3$. Let $g \in G$, and let $x \in N$. Then $gx = gxg^{-1} \cdot g \in Ng$. Thus, we have $gN \subset Ng$. In the same manner, we get $Ng \subset gN$.
Finally we prove $3 \Rightarrow 1$. Let $g \in G$ and $x \in N$. We want to show that $gxg^{-1} \in N$, i.e., that $gx \in Ng$. Since $gx \in gN$ which is equal to Ng by assumption, we are done. □

Corollary 1

If group G has only one subgroup H of size r, then H is normal. Indeed, for any $g \in G$ the set gHg^{-1} is also a subgroup of G. Since the map

$$H \to gHg^{-1} \qquad x \mapsto gxg^{-1}$$

is a bijection (see homework # 9), gHg^{-1} is also a subgroup of size r. By assumption, such subgroup is unique, so

$$gHg^{-1} = H,$$

implying via part 2 of the proposition that H is normal.

Corollary 2: S

bgroup $H \subset G$ of index 2 is normal.

Proof. Indeed, if $H \subset G$ has index two, then the left/right cosets of H are H and $G \backslash H$. Since the left and right cosets coincide, part 3 pf Proposition 1 implies that H is normal. □

It is useful to keep in mind also non-examples.

Example 2: Non-example

Subgroup $H = \{\mathrm{id}, (12)\} \subset S_3$ is not normal. Indeed, the left coset generated by (23) is

$$(23)H = \{(23), (23)(12)\} = \{(23), (132)\},$$

while the right coset is

$$H(23) = \{(23), (12)(23)\} = \{(23), (123)\}.$$