Lecture 2

Reference: Judson, Chapters 1 & 2

Maps

Definition 1: Inverse map

Map $g\colon B \to A$ is the **inverse** of the map $f\colon A \to B$ if

$$(g \circ f) = \mathrm{id}_A, \text{ and } (f \circ g) = \mathrm{id}_B,$$

where id_A and id_B are the **identity maps** of sets A and B respectively. The inverse map is usulay denoted by

$$f^{-1}\colon B \to A.$$

Example 1

Map

$$g\colon [0, +\infty) \to [0, +\infty), \quad g(x) = \sqrt{x}$$

is the inverse of the map

$$f\colon [0, +\infty) \to [0, +\infty), \quad f(x) = x^2$$

At the same time map

$$f\colon (-\infty, +\infty) \to [0; +\infty), \quad f(x) = x^2$$

does not admit an inverse[a].

[a] Why?

When does a map admits an inverse? The following theorem answers this question.

Theorem 1

Map $f\colon A \to B$ **admits an inverse** if and only if f is **bijective**. In this case the inverse is unique.

NB: Whenever we encounter **if and only if** claim, a proof in **two directions** is required!

Proof. Direction $\Longrightarrow$.
Assume that f admits an inverse $g\colon B \to A$. We are about to prove that f is injective and bijective.

- (f is injective). Indeed, assume that $f(x) = f(y)$ for some $x, y \in A$. Apply map g to both sides of this identity, then
$$x = \mathrm{id}_A(x) = (g \circ f)(x) = g(f(x)) = g(f(y)) = (g \circ f)(y) = \mathrm{id}_A(y) = y.$$
So f is injective.
- (f is surjective). Take any $b \in B$. Then
$$f(g(b)) = (f \circ g)(b) = \mathrm{id}_B(b) = b,$$
so b is in the range of f, and therefore f is surjective.

Direction $\Longleftarrow$.
Assume that f is both injective and surjective. We are about to construct function $g\colon B \to A$ which will be the inverse of f. Given any $b \in B$ we can find $a \in A$ such that $f(a) = b$, since f is surjective. Then we define

$$g(b) = a.$$

It remains to check that g is indeed the inverse map.

- Check that $(f \circ g) = \mathrm{id}_B$. Take any $b \in B$. Then by definition of g, we have $f(g(b)) = b$. Therefore indeed $(f \circ g)(b) = b$.
- Check that $(g \circ f) = \mathrm{id}_A$. Take any $a \in A$. Then we want to check that

$$(g \circ f)(a) = a.$$

We already know that

$$f(g(f(a))) = f(a),$$

since $f \circ g = \mathrm{id}_B$. Now, as f is injective the latter implies

$$g(f(a)) = a,$$

as required.

□

Equivalence relations

Often, given a set A, we would like to consider a *coarser* set by identifying some of the elements of A with each other. For example if $A = \{\text{days in October}\}$, then planning a schedule, we might want to identify the same days of the week, e.g., October 6 would be the same to us as October 13, October 20, October 27. This procedure is formalized through the notion of *equivalence relations*.

Definition 2: Equivalence relation

Equivalence relation on a set A is a subset $R \subset A \times A$ satisfying the following properties

- (reflexive) for any $a \in A$ we have $(a,a) \in R$
- (symmetric) if $(a,b) \in R$, then $(b,a) \in R$.
- (transitive) if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

If we have $(a,b) \in R$ we will write '$a \sim b$' and say 'a is equivalent to b'.

Set $R \subset A \times A$ consists of all pairs $a, b \in A$ which we would like to identify with each other.

Example 2

1. $R = \{(a,a) \mid a \in A\}$. In other words, any element a is equivalent only to itself.
2. $R = \{(a,b) \mid a,b \in A\}$. In other words, any element $a \in A$ is equivalent to any other element $b \in A$.
3. Take $R = \mathbb{Z}$ and define $R = \{(m,n) \mid m-n \text{ is even}\} \subset \mathbb{Z} \times \mathbb{Z}$.

If $a \in A$ and $\sim$ is an equivalence relation, we can form an equivalence class

$$[a] = \{b \in A \mid b \sim a\}.$$

Proposition 1

Let $\sim$ be an equivalence relation on the set A. Then any two equivalence classes $[a]$, $[b]$ either coincide element-wise, or do not intersect.

Proof. Exercise. □

The above exercise allows to define a new set consisting of the equivalence classes:

Definition 3

The set of *equivalence classes in A* ***modulo*** *an equivalence relation* $\sim$ is defined as

$$A/\sim \,=\, \{[a] \mid a \in A\}.$$

There is a natural map

$$A \to A/\sim, \quad a \mapsto [a]$$

sending each element to its equivalence class.

Problem 1: When is the natural map $A \to A/\sim$ surjective? injective? bijective?

Example 3

Going back to Example 2, we see that
If $R = \{(a,a) \mid a \in A\}$ then $A/\sim$ is the same as A
If $R = \{(a,b) \mid a, b \in A\}$ then $A/\sim$ has only one element
Finally, in the last example $\mathbb{Z}/\sim$ consists of two elements: class of odd integers and class of even integers.