Lecture 20

Reference: Judson, Chapter 11

First isomorphism theorem

Let $f: G \to H$ be a homomorphism. Then we have a normal subgroup

$$\mathrm{Ker}(f) \subset G,$$

thus we can also define the *quotient group*

$$G/\mathrm{Ker}(f).$$

One might ask

Question 1

Is there a relation between the groups $G/\mathrm{Ker}(f)$ and H and homomorphism H?

The answer to this question is given by the following fundamental result.

Theorem 1: First Isomorphism Theorem

Let $f: G \to H$ be a homomorphism. Let

$$\pi: G \to G/\mathrm{Ker}(f)$$

be the natural surjective homomorphism sending every element $x \in G$ to the coset containing x. Then there exists a unique isomorphism

$$\widetilde{f}: G/\mathrm{Ker}(f) \to \mathrm{Im}(f) \subset H$$

such that for any $x \in G$

$$f(x) = (\widetilde{f} \circ \pi)(x).$$

Proof. From the previous lecture we know that cosets of the kernel $\mathrm{Ker}(f) \subset G$ are in 1-to-1 correspondence with the elements in $\mathrm{Im}(f)$. Thus we have a well-defined bijection

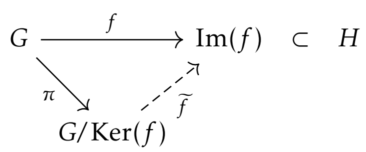
$$\widetilde{f}: G/\mathrm{Ker}(f) \to \mathrm{Im}(f),$$

which sends a coset $A \subset G$ to $f(a) \in \mathrm{Im}(a)$, where a is any representative of A.
We need to check that $\widetilde{f}$ is a homomorphism. Indeed, if $x\mathrm{Ker}(f)$ and $y\mathrm{Ker}(f)$ are any two elements of $G/\mathrm{Ker}(f)$, then

$$\widetilde{f}(x\mathrm{Ker}(f) \cdot y\mathrm{Ker}(f)) =^1 \widetilde{f}(xy\mathrm{Ker}\, f) =^2 f(xy) =^3 f(x)f(y) =^4 \widetilde{f}(x\mathrm{Ker}(f))\widetilde{f}(y\mathrm{Ker}(f)),$$

where in the first identity we used the definition of the multiplication in the quotient group $G/\mathrm{Ker}(f)$, in the second identity we used the definition of $\widetilde{f}$, in the identity 3 we used the fact that f is a homomorphism, and in the last identity 4 we again used the definition of $\widetilde{f}$. □

Alternatively, we can formulate the First Isomorphism Theorem via a *commutative* diagram:



which says that given a homomorphism $f: G \to H$ and the corresponding surjective homomorphism $\pi: G \to G/\mathrm{Ker}(f)$, there exists a unique isomorphism $\widetilde{f}: G/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ such that the above diagram is *commutative*, i.e., $f = \widetilde{f} \circ \pi$.

Remark 1

In many situations, the first isomorphism theorem allows us to get a better understanding of the quotient groups G/N. Specifically, if we want to describe G/N explicitly, often it is helpful to find a homomorphism

$$f: G \to H$$

to some *other* group H, such that $\mathrm{Ker}(f) = N$. Then, by the theorem G/N is isomorphic to $\mathrm{Im}(f)$.

Example 1

1. Consider $G = S_n$ and its normal subgroup $N = A_n$. Then we can realize A_n as the kernel of the sign homomorphism

$$\mathrm{sgn}\colon S_n \to \{+1, -1\}.$$

Thus

$$A_n/S_n = \mathrm{Im}(\mathrm{sgn}) = \{+1, -1\},$$

and the latter group is isomorphic to $\mathbb{Z}_2$.

2. Let $G = GL_2(\mathbb{R})$ be the group of 2×2 matrices with nonzero determinant. Consider its subgroup

$$SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}) \mid \det(A) = 1\}.$$

This group is the kernel of the determinant homomorphism

$$\det\colon GL_2(\mathbb{R}) \to \mathbb{R}\backslash\{0\}.$$

Thus, since det is surjective (i.e., $\mathrm{Im}(\det) = \mathbb{R}\backslash\{0\}$), we deduce from the isomorphism theorem

$$GL_2(\mathbb{R})/SL_2(\mathbb{R}) \simeq \mathbb{R}\backslash\{0\}.$$