Midterm

This is a **closed book** exam. To get the full credit, write complete, detailed solutions. No credit will be given for an answer without a proof.

Problem 1 (10 points). *a*) Let H be a group. What does it mean for H to be *non-commutative*?

b) Let G be a commutative group, and H a *non-commutative* group. Prove that there is no injective homomorphism

$$\varphi\colon H \to G.$$

Problem 2 (10 points). Consider group[1] $G = \mathbb{Z}\times\mathbb{Z}_3$.

a) Find all elements $g \in G$ of finite order.

b) Find an *infinite* proper subgroup $H \subsetneq G$.

Problem 3 (10 points). Prove or disprove: for any of the following statements, figure out if it is true or false, by either proving it or providing a counterexample.

a) Group $\mathbb{Z}_{100}$ has a subgroup of order $d \in \mathbb{Z}$ for every divisor d of 100.

b) Set H consisting of matrices

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

is a group with respect to matrix multiplication and is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$.

Problem 4 (10 points). Consider a group $G = \mathbb{Z}\times\mathbb{Z}$ and let H to be a subset

$$H = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid x \text{ and } y \text{ are even}\} \subset H$$

a) Prove that H is a subgroup.

b) How many left cosets gH are there?

Problem 5 (10 points). Let p be a prime number.

a) How many multiples of p are there in the set $\{1,2,\ldots,p^k\}$?

b) Find the size of the group $(\mathbb{Z}_{p^k}^\times, \times)$.

Problem 6 (10 points). *a*) Find three non-isomorphic groups of order 8.

(Make sure to prove that the groups are indeed non-isomorphic.)

b) Find a subgroup of order 2 in each of these groups.

[1] As usual, groups $\mathbb{Z}$ and $\mathbb{Z}_n$ are considered with respect to addition.