Homework 1

Due: Tuesday, September 17

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You can use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. Let $z_0 = x + \boldsymbol{i}y \in \mathbb{C}$ be a number such that $x \neq 0$. Prove that the function $\arctan(y/x)$ is harmonic by finding a holomorphic function $f(z)$ in a neighbourhood of z_0 such that $\operatorname{Im} f(z) = \arctan(y/x)$.

Problem 2. Draw on the complex plane all solutions of the equation $z^3 = -\boldsymbol{i}$.

Problem 3. *a*) Let z, w be two complex numbers such that $z\overline{w} \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1, \qquad \text{if } |z| < 1 \text{ and } |w| < 1$$

and

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1, \qquad \text{if } |z| = 1 \text{ or } |w| = 1$$

b) Prove that for a fixed $w \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ the mapping

$$F_w \colon z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions

- F_w is holomorphic and maps $\mathbb{D}$ to $\mathbb{D}$.
- $F_w(0) = w$ and $F_w(w) = 0$
- $|F_w(z)| = 1$ if $|z| = 1$.
- $F_w \colon \mathbb{D} \to \mathbb{D}$ is bijective.

Problem 4. Consider the function

$$f(x + \boldsymbol{i}y) = \sqrt{|x||y|}, \qquad x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at $(0,0)$ but f is not holomorphic at this point. Why there is no contradiction with Theorem 13 from Lecture 2?

Problem 5. Prove that

a) The power series $\sum nz^n$ does not converge anywhere on the unit circle.

b) The power series $\sum z^n/n^2$ converges everywhere on the unit circle.