Homework 2

Due: Tuesday, September 24

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. *a)* Find the imaginary and real parts of $(1+\boldsymbol{i})^{\boldsymbol{i}} := e^{\operatorname{Log}(1+\boldsymbol{i})\boldsymbol{i}}$. *b)* Show that the set of values of $\log(\boldsymbol{i}^2)$ is not the same as the set of values of $2\cdot\log\boldsymbol{i}$. *c)* Show that $\operatorname{Log}(-1+\boldsymbol{i})+\operatorname{Log}(-1+\boldsymbol{i})\neq\operatorname{Log}((-1+\boldsymbol{i})^2)$.

The above problem shows that one should be cautious working with complex logarithms!

Problem 2. Assume that $f\colon\mathbb{C}\to\mathbb{C}$ is holomorphic with $|f(z)|=1$ everywhere. Prove that f is constant.

Hint: given $f(x+\boldsymbol{i}y)=u+\boldsymbol{i}v$, differentiate u^2+v^2 with respect to x and y and use Cauchy-Riemann equations to deduce that u/v and v/u are constant whenever defined.

Problem 3. Given $z=x+\boldsymbol{i}y$, we define trigonometric functions

$$\cos z := \frac{e^{\boldsymbol{i}z}+e^{-\boldsymbol{i}z}}{2} \qquad \sin z := \frac{e^{\boldsymbol{i}z}-e^{-\boldsymbol{i}z}}{2\boldsymbol{i}}$$

and their hyperbolic counterparts

$$\cosh z := \frac{e^{z}+e^{-z}}{2} \qquad \sinh z := \frac{e^{z}-e^{-z}}{2}.$$

Prove that *a)* $|\cos z|^2=\cos^2 x+\sinh^2 y$ and $|\sin z|^2=\sin^2 x+\sinh^2 y$; *b)* $|\cos z|^2+|\sin z|^2\geqslant 1$ with equality if and only if z is purely real.

Problem 4. Evaluate

$$\int_\gamma \left(\frac{1}{z}+\boldsymbol{i}\right)^2 dz,$$

where γ is the straight segment connecting $\boldsymbol{i}$ and $2\boldsymbol{i}$.

Problem 5. Prove that the function $f(z)=\frac{1}{z-1}-\frac{1}{z+1}$ is *a)* holomorphic in $\mathbb{C}-\{-1,1\}$; *b)* does **not** have a primitive in $\mathbb{C}-\{-1,1\}$; *c)* does have a primitive in $\mathbb{C}-\{t+\boldsymbol{i}0 \mid t\in[-1,1]\}$.

Hint: $\frac{z-1}{z+1}\notin(-\infty,0]$ for $z\in\mathbb{C}-\{t+\boldsymbol{i}0\mid t\in[-1,1]\}$. Hence function $F(z):=\operatorname{Log}\frac{z-1}{z+1}$ is a well-defined holomorphic, single-valued function on $\mathbb{C}-\{t+\boldsymbol{i}0\mid t\in[-1,1]\}$.