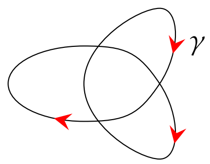
Homework 3

Due: Tuesday, October 1

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. Given γ as below, find all possible values of the integral $\int\limits_{\gamma} \frac{dz}{z-z_0}$, where $z_0 \in \mathbb{C}$, $z_0 \notin \gamma$.



Problem 2. Prove that a function which is holomorphic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial. What can be the degree of such polynomial?

Problem 3. Compute integrals for $n, m \in \mathbb{Z}$

a) $\int\limits_{|z|=1} e^z z^{-n} dz$;[1]

b) $\int\limits_{|z-2|=3/2} z^n (1-z)^m dz$.

Problem 4. Prove that for any holomorphic function $\mathbb{D} \to \mathbb{C}$ and $r \in (0,1)$ we have

$$2f'(0) = \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2} d\zeta.$$

Hint: function $F(\zeta) = \frac{f(\zeta)-f(-\zeta)}{\zeta}$ extends to a holomorphic function in $\mathbb{D}$ with $F(0) = 2f'(0)$.

Problem 5. For a holomorphic function $f \colon \mathbb{D} \to \mathbb{C}$ define

$$d := \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|.$$

Prove that $2f'(0) \leqslant d$.

Hint: use Problem 4.

Bonus question[2] Let $f(z)$ be as in Problem 5. Prove that the equality holds if and only if $f(z)$ is linear: $f(z) = a_0 + a_1 z$.

[1] Unless specified otherwise, all circular contours are traveled counterclockwise.
[2] You are not expected to solve this problem and no credit will be given.