Homework 4

Due: Tuesday, October 8

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. Does there exist a holomorphic function $f(z)$ in the unit disk such that successive derivatives of $f(z)$ satisfy $\left|\frac{f^{(n)}(0)}{n!}\right| = 2^n$?

Problem 2. Does there exists a nonzero holomorphic function $f(z)$ in the unit disk $\mathbb{D}$ and a sequence of points $a_n \in \mathbb{D}$ such that $f(a_n) = 0$ and a_n converge to a point in the closed disk $a \in \overline{\mathbb{D}}$?

Problem 3. Find the power series representing function $f(z) = 1/z$ in a neighbourhood of point $1 + \boldsymbol{i} \in \mathbb{C}$. What is the radius of convergence of this power series?

Problem 4. Consider holomorphic functions $f(z)$ and $g(z)$ with a zero of orders n and m respectively at $a \in \mathbb{C}$. What are the possible values of the order of zero at a for $a)$ $f(z) - g(z)$; $b)$ $f(z) \cdot g(z)$?

Problem 5. Prove that function

$$f(z) = \frac{e^z - e^{-z}}{2z} - 1$$

has a removable singularity at $z = 0$ with $f(0) = 0$ and find the order of zero at $z = 0$.