Homework 5

Due: Tuesday, October 15

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. For $R \in (0, +\infty)$, such that $e^z - 1$ is nonzero on the circle $|z| = R$, evaluate the integral

$$\int_{|z|=R} \frac{e^z}{e^z - 1} dz.$$

Problem 2. Function $f(z)$ is holomorphic in $\{\mathrm{Im}(z) > 0\}$ and bounded by M. Find an upper bound for $f^{(n)}(z)$ in $\{\mathrm{Im}(z) > r\}$.

Problem 3. Let $U \subset \mathbb{C}$ be a *bounded* neighbourhood of $0 \in \mathbb{C}$. Consider a holomorphic function $f\colon U \to U$. Assume that $f(0) = 0$ and $f'(0) = 1$. Prove that $f(z) = z$.

Hint: Write $f(z) = z + a_n z^n + O(z^{n+1})$ near 0 and show that k-fold composition $f_k := f \circ \cdots \circ f$ satisfies $f_k(z) = z + k a_n z^n + O(z^{n+1})$. Use Cauchy's inequalities with $k \to \infty$ to conclude that $a_n = 0$.

Problem 4. Find the number of zeros of $f(z) = z^5 + 3z - 1$ in the annulus $\{1 < |z| < 2\}$.

Problem 5. Prove that if $f(z)$ is an *injective* entire function, then $f(z) = az + b$ with $a \neq 0$[1].

Hint: Use open mapping theorem and Casorati-Weierstrass to prove that $f(z)$ cannot have essential singularity at ∞.

[1] This problem shows that the group holomorphic isomorphisms of $\mathbb{C}$ is isomorphic to the group of affine transformations $f(z) = az + b$.