Homework 6

Due: Tuesday, October 29

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. Show that a holomorphic function $f(z)\colon \mathbb{D} \to \mathbb{D}$ satisfies

$$\frac{|f'(z)|}{(1-|f(z)|^2)} \leqslant \frac{1}{1-|z|^2}$$

Hint: use general form of Schwarz lemma and let $z_1 \to z_2$.

Problem 2. Find the poles and residues of $a)\ f(z) = \dfrac{1}{\sin z}$; $b)\ f(z) = \dfrac{1}{z^m(1-z)^n}$, where $m, n \in \mathbb{N}$.

Problem 3. Evaluate the integrals

$$a) \int\limits_0^{+\infty} \frac{\cos x}{x^2+a^2} dx \quad a \in \mathbb{R}; \qquad b) \int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^4}$$

Problem 4. Let $R(x) = P(x)/Q(x)$ be a rational function such that $Q(x)$ does not have real roots and $\deg Q \geqslant \deg P + 2$. Prove that

$$\int_{-\infty}^{+\infty} R(x)dx = 2\pi \boldsymbol{i} \sum_{\operatorname{Im} z_k > 0} \operatorname{res}_{z_k} R(z).$$