Homework 7

Due: Tuesday, November 5

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

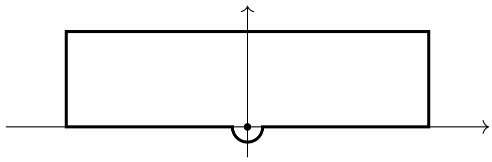
Problem 1. Find conjugate harmonic functions for the following functions *a)* $u(x,y) = e^y \cos x$ in $\mathbb{C}$ *b)* $u(x,y) = \log(x^2 + y^2)$ in $\{x > 0\}$.

Problem 2. Prove that if $f: z \in U \to \mathbb{C}$ is holomorphic and $u: W \to \mathbb{R}$ is harmonic in a neighbourhood W of the range of f, then $(u \circ f)(z) := u(\mathfrak{Re} f(z), \mathfrak{Im} f(z))$ is harmonic in U.

Problem 3. Evaluate the integral

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx.$$

Hint: Apply residue theorem to the function $f(z) = \frac{e^{iz}}{z}$ and the contour γ of the following shape:



Problem 4 (Poisson's formula for the half plane)**.** Assume that function $U(x)$ is piecewise continuous and bounded for $x \in \mathbb{R}$. *a)* Show that

$$P_U(z) := \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi$$

represents a harmonic function in the upper half plane.

b) Prove that $P_U(z)$ coincides with $U(x)$ at points of continuity.

Hint: function $f(z) = \frac{z-i}{z+i}$ maps the upper half-plane bijectively onto the unit disk $\mathbb{D}$.

Problem 5. Given a complex-valued continuous function $\varphi: \{|z| = 1\} \to \mathbb{C}$ does there always exist a continuous function $f: \overline{\mathbb{D}} \to \mathbb{C}$, holomorphic in $\mathbb{D}$ which coincides with $\varphi(z)$ on $\{|z| = 1\}$?