# Homework 8

**Due:** Tuesday, November 12

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

**Problem 1.** Prove that for $|z| < 1$

$$(1+z)(1+z^2)(1+z^4)(1+z^8)\cdots = \frac{1}{1-z}.$$

**Problem 2.** Prove that

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

converges absolutely and uniformly on every compact set.

**Problem 3.** Show that the Laurent series of $f(z) = 1/(e^z - 1)$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{i=1}^{\infty} \frac{B_k}{(2k)!} z^{2k-1}, \quad B_k \in \mathbb{R}.$$

Number $B_k$ are called *Bernoulli numbers*. Calculate (with proof!) $B_1$ and $B_2$.

**Problem 4.** Prove the identity

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{-\infty}^{+\infty} \frac{1}{(z-n)^2}.$$

**Problem 5.** Comparing coefficients in the Laurent series of $\cot \pi z$ (or $\pi^2/\sin^2 \pi z$) and of its expression as a sum of partial fractions, find the values of

$$\sum \frac{1}{n^2} \text{ and } \sum \frac{1}{n^4}.$$