Homework 9

Due: Tuesday, November 19

Each problem is worth 10 points. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. Prove the reflection formula for $\Gamma(s)$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Use this identity to find $\Gamma(1/2)$.

Problem 2. Prove that for all complex numbers $s \neq 0, -1, -2, \ldots$

$$\frac{d^2 \log \Gamma(s)}{ds^2} = \sum_{n=0}^{\infty} \frac{1}{(s+n)^2}$$

where $\frac{d^2 \log \Gamma(s)}{ds^2}$ can be defined as $\left(\frac{\Gamma'(s)}{\Gamma(s)}\right)'$.

Problem 3 (Legendre's Duplication Formula)**.** *a)* Prove that $\Gamma(z)\Gamma(z+\frac{1}{2})$ and $\Gamma(2z)$ have the same poles. *b)* Prove that $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$

Hint: Weierstrass factorizations of functions $(\Gamma(z)\Gamma(z+\frac{1}{2}))^{-1}$ and $\Gamma(2z)^{-1}$ must differ by a factor e^{az+b}.

Problem 4. Prove that for $\mathfrak{Re}(s) > 1$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$

Hint: Write $1/(e^x-1) = \sum_{n=1}^{\infty} e^{-nx}$.

Problem 5. Let $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ be the Möbius function

$$\mu(n) = \begin{cases} (-1)^d, & \text{if } n \text{ is square-free and has } d \text{ prime factors} \\ 0, & \text{otherwise} \end{cases}$$

Prove that for $\mathfrak{Re}(s) > 1$

$$\zeta(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$