Lecture 1

Foreword

[1] Complex analysis is one of the most beautiful fields of mathematics. It has numerous connections with the most of modern branches of pure and applied mathematics: algebraic geometry, number theory, physics (electrostatics, hydrodynamics, heat conduction), probability, combinatorics.

Historically, complex numbers were introduced in 16th century as a way to interpret the *Cardano formula* for the roots of cubic polynomial $x^3 + px + q = 0$:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

To obtain all 3 roots of the cubic equation we have to interpret the cubic root as a multivalued function with values in $\mathbb{C}$.

For several hundred years after Cardano complex numbers remained an obscure topic. It took several centuries and efforts of the best mathematicians of their time (Euler, Gauss, Weierstrass, Schwarz, Cauchy, Abel and many others) to demonstrate the fascinating nature of complex numbers and complex analysis, and build the ground for modern applications.

Field $\mathbb{C}$ of Complex Numbers

Definitions

Let $\boldsymbol{i}$ denote the *imaginary unit* defined by the property

$$\boldsymbol{i}^2 = -1.$$

Sometimes element $\boldsymbol{i}$ is called *the square root of* -1 and denoted by $\sqrt{-1}$. We define the complex numbers $\mathbb{C}$ by adjoining element $\boldsymbol{i}$ to the field of real numbers:

Definition 1. *Complex numbers* $\mathbb{C} := \mathbb{R}[\boldsymbol{i}]$ is the vector space over $\mathbb{R}$ spanned by 1 and $\boldsymbol{i}$, i.e., it is the set of all linear combinations

$$a + \boldsymbol{i}b, \quad a, b \in \mathbb{R}.$$

a and b are called *real* and *imaginary* parts of a complex number $z = a + \boldsymbol{i}b$:

$$a = \mathfrak{Re}(z), b = \mathfrak{Im}(z).$$

Being a vector space, $\mathbb{C}$ is closed under addition:

$$(a + \boldsymbol{i}b) + (c + \boldsymbol{i}d) = (a + c) + \boldsymbol{i}(b + d).$$

Moreover, using the defining property of $\boldsymbol{i}$ there is a unique way to turn $\mathbb{C}$ into a *commutative ring* by defining multiplication as

$$(a + \boldsymbol{i}b)(c + \boldsymbol{i}d) = ac + \boldsymbol{i}ad + \boldsymbol{i}bc + \boldsymbol{i}^2bd = (ac - bd) + \boldsymbol{i}(ad + bc).$$

Complex numbers of the form $z = a + \boldsymbol{i}0$ are called *purely real* and are abbreviated as $z = a$. The neutral elements in $\mathbb{C}$ with respect to addition is $z = 0$, and the neutral element with respect to multiplication is $z = 1$.

The key feature of $\mathbb{C}$ is that it is also a *field*, i.e., we can not only add up, subtract and multiply complex numbers, but also can divide by nonzero elements:

$$\frac{1}{a + \boldsymbol{i}b} = \frac{a - \boldsymbol{i}b}{(a + \boldsymbol{i}b)(a - \boldsymbol{i}b)} = \frac{a - \boldsymbol{i}b}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \boldsymbol{i}\left(-\frac{b}{a^2 + b^2}\right).$$

By the very definition of $\mathbb{C}$, the quadratic polynomial $x^2 + 1$, which is irreducible over $\mathbb{R}$, now has two roots over $\mathbb{C}$: $x = \pm\boldsymbol{i}$.

[1] Large part of these notes is borrowed from the lectures by Antoine Cerfon.

Exercise 1. Prove that any quadratic polynomial with real coefficients has two complex roots (counting with multiplicities).

Theorem 2 (Fundamental Theorem of Algebra). *Any polynomial $p(x) \in \mathbb{C}[x]$ of degree $n \geqslant 1$ with complex coefficient can be factored as*

$$p(x) = a_n(x - z_1)(x - z_2)\dots(x - z_n),$$

i.e., $p(x)$ has exactly n roots over $\mathbb{C}$ (with multiplicities).

There exist many proofs o the Fundamental Theorem of Algebra, but the quickest and the most beautiful proofs can be obtained with the use of complex analysis. We will discuss three of them later in this course.

Exercise 2. Let $M \subset \mathrm{Mat}_2(\mathbb{C})$ be the set of 2×2 matrices:

$$M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

(a) Prove that M is closed under matrix addition and multiplication $(+, \times)$.

(b) Show that $(M, +, \times)$ is isomorphic to the field of complex numbers.

Conjugation

In computing the multiplicative inverse of $z = a + \boldsymbol{i}b$ we used the *complex conjugate* of z defined as

$$\bar{z} := a - \boldsymbol{i}b$$

Conjugation satisfies several important properties:

- conjugations is involutive: $\bar{\bar{z}} = z$
- $z \cdot \bar{z}$ is a positive real number, unless $z = 0$.
- conjugation is an additive and multiplicative automorphism of $\mathbb{C}$:

  $\overline{z + w} = \bar{z} + \overline{w}$

  $\overline{z \cdot w} = \bar{z} \cdot \overline{w}$

Remark 3. As a corollary of the last property we conclude that if $w \in \mathbb{C}$ is a root of a polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ with real coefficients, i.e., $p(w) = 0$, then $\overline{w}$ is also a root:

$$p(\overline{w}) = \overline{w}^n + a_{n-1}\overline{w}^{n-1} + \dots + a_1\overline{w} + a_0 = \overline{w^n + a_{n-1}w^{n-1} + \dots + a_1 w + a_0} = 0.$$

This observation implies that all complex, not purely real roots a polynomial with real coefficients come in conjugate pairs: w and $\overline{w}$. Combining this observation with the Fundamental Theorem of Algebra we conclude that any polynomial with real coefficients of degree $n \geqslant 1$ can be factored as a product of linear and quadratic factors:

$$p(x) = a_n \prod_i (x - x_i) \cdot \prod (x^2 + p_i x + q_i).$$

We can express the real and imaginary parts of $z = a + \boldsymbol{i}b$ in terms of conjugation:

$$\mathfrak{Re}(z) = \frac{z + \bar{z}}{2} \qquad \mathfrak{Im}(z) = \frac{z - \bar{z}}{2\boldsymbol{i}}$$

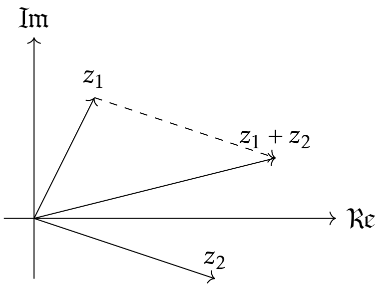
We define *modulus* $|z|$ of $z = a + \boldsymbol{i}b$ by the identity $|z|^2 = z\bar{z}$. Clearly $|z| = \sqrt{a^2 + b^2}$. Properties of conjugation imply that

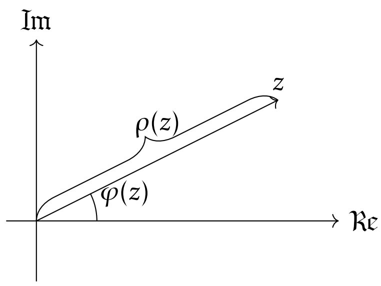
$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|,$$

i.e., $|\cdot| \colon \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ is a multiplicative homomorphism.

Complex plane

It is convenient to picture complex numbers $z = a + \boldsymbol{i}b$ as points (or zero-rooted vectors) on the Cartesian plane $\mathbb{R}^2$ with coordinate axes $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$. This way the modulus of $z = a + \boldsymbol{i}b$ has a clear interpretation as the Euclidean length $\sqrt{a^2 + b^2}$ of the corresponding vector. Addition of complex numbers translates into the addition of the underlying vectors.





Instead of Cartesian coordinates $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$ on $\mathbb{R}^2$ we can consider *polar coordinates* $(\rho(z), \varphi(z))$ such that for $z = a + \boldsymbol{i}b$ we have

$$a = \rho\cos\varphi, \quad b = \rho\sin\varphi$$

$$z = \rho(\cos\varphi + \boldsymbol{i}\sin\varphi).$$

Number $\rho(z) = |z|$ is the *modulus* of a complex number and number $\varphi(x) \in [0, 2\pi)$ is the *argument*. Of course the choice of the range for the argument function is somewhat arbitrary and could be chosen to be, e.g., $(-\pi, \pi]$ instead.

Exercise 3. Prove that for angles φ and ψ we have

$$(\cos\varphi + \boldsymbol{i}\sin\varphi)(\cos\psi + \boldsymbol{i}\sin\psi) = \cos(\varphi + \psi) + \boldsymbol{i}\sin(\varphi + \psi).$$

The above exercise "follows" from the celebrated *Euler's formula*:

$$e^{\boldsymbol{i}\varphi} := (\cos\varphi + \boldsymbol{i}\sin\varphi)$$

At the moment, this formula does not make sense, since we have not defined exponent of a complex number. We will make it precise later in the course by, first defining the left-hand side, and then using the Euler's formula as the definition of cos and sin functions. For now, we will use $e^{\boldsymbol{i}\varphi}$ as a shorthand for the right hand side of Euler's formula.

Topology on $\mathbb{C}$

Complex plane has a natural *topology*, i.e., a collection of *open subsets* $U \subset \mathbb{C}$:

Definition 4. Subset $U \subset \mathbb{C}$ is called *open* if for any $z \in U$ there exists $\epsilon > 0$ such that the open ball $B_\epsilon(z) := \{w \in \mathbb{C} \mid |z - w| < \epsilon\}$ is contained in U.

Now, once we have endowed $\mathbb{C}$ with a structure of a topological space, we can define continuous functions on $\mathbb{C}$. We will make it explicit in our next class.

Polar coordinates and multiplication

If $z_1 = \rho_1 e^{\boldsymbol{i}\varphi_1}$ and $z_2 = \rho_2 e^{\boldsymbol{i}\varphi_2}$ then $z_1 z_2 = \rho_1\rho_2 e^{\boldsymbol{i}(\varphi_1+\varphi_2)}$, i.e., under multiplication of two complex numbers their arguments add up and their moduli multiply. A particularly useful consequence of this fact is *de Moivre's formula*. For $n \in \mathbb{Z}$:

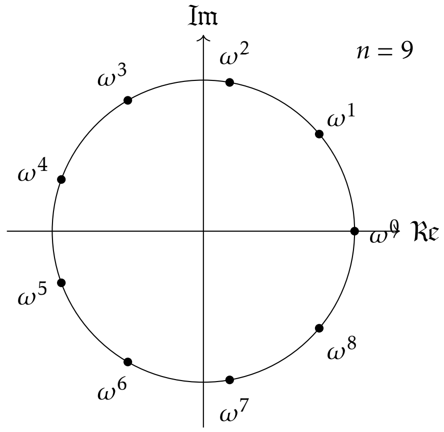
$$(\cos\varphi + \boldsymbol{i}\sin\varphi)^n = \cos(n\varphi) + \boldsymbol{i}\sin(n\varphi)$$

Using de Moivre's formula we can also find nth root of a complex number z. Namely, if $z = \rho e^{\boldsymbol{i}\varphi}$, then there are precisely n numbers

$$w = \rho^{1/n} e^{\boldsymbol{i}\left(\frac{\varphi}{n} + \frac{2\pi k}{n}\right)}, \quad k = 0, \ldots, n-1$$

satisfying $w^n = z$. A particular important case is $z = 1$ which yields n roots of unity:

$$1, \omega, \omega^2, \ldots, \omega^{n-1}, \quad \omega := e^{\boldsymbol{i}\frac{2\pi}{n}}.$$



Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

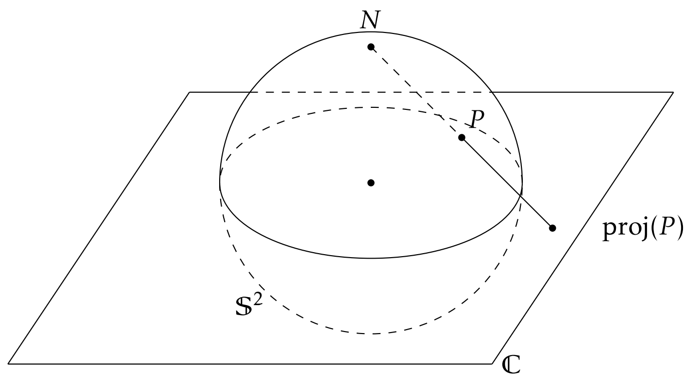
Instead of the complex plane $\mathbb{C}$, we will often need to consider the *one-point compactification* of $\mathbb{C}$.

Definition 5. *Riemann sphere* $\widehat{\mathbb{C}}$ as a set, is the complex plane $\mathbb{C}$ together with a point ∞ at "infinity". We turn $\widehat{\mathbb{C}}$ into a topological space by taking as open sets all the open subsets U of $\mathbb{C}$ together with $V = (\mathbb{C} \backslash K) \cup \{\infty\}$, where $K \subset \mathbb{C}$ is *compact* (closed and bounded).

We can cover $\widehat{\mathbb{C}}$ with two *charts* $U_1 \simeq \mathbb{C}$ and $U_2 \simeq \mathbb{C}$ such that $U_1 \cap U_2 = \mathbb{C}\backslash\{0\}$ and gluing them along the maps

$$\mathbb{C} \leftarrow \mathbb{C}\backslash\{0\} \rightarrow \mathbb{C}$$
$$z \leftarrow\!\shortmid z \mapsto z^{-1}.$$

The geometric intuition for considering the Riemannian sphere is provided by *stereographic projection* which we 'define' using a picture:



Namely, given a unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ which intersects the complex plane $\{z = 0\}$ in the unit circle we project any point $P \in \mathbb{S}^2$ on the sphere onto the plane from the north pole N. Projection $\mathrm{proj}\colon \mathbb{S}^2\backslash\{N\} \to \mathbb{C}$ is well-defined everywhere except for the north pole N. By adding a point ∞ to the complex plane, and setting $\mathrm{proj}(N) = \infty$, we extend stereographic projection to a continuous bijective map between $\mathbb{S}^2$ and $\widehat{\mathbb{C}}$.

Exercise 4. Prove that for any $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$ the function

$$f(z) = \frac{az + b}{cz + d}$$

extends to a well-defined bijective continuous map

$$\widehat{f}\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}.$$

Transformations of this type form a group called *Möbius group*.

Exercise 5. Prove that Möbius group is isomorphic to $\mathrm{PSL}_2(\mathbb{C}) := \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{Id}\}$.