Lecture 10

Argument principle

Let $f(z)$ ba a not identically zero holomorphic function in an open disk D. Often it is important to locate and count zeros of f in D. The *argument principle* is a powerful tool solving this problem.

Let $\gamma \subset D$ be a contour such that $f(z) \neq 0$ on γ. Curve γ is contained in an open disk $D' \subsetneq D$ and function $f(z)$ can only have finitely zeros in D'. Let Z=$\{\zeta_i\}_{j=1}^N$ be the set of zeros of f in D' counted with multiplicities (i.e., if ζ is a zero of order k, then it appears k times in Z).

Theorem 1 (Argument principle)**.** *Under the above assumptions,*

$$\sum_{j=1}^{N} n(\gamma, \zeta_j) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f'(z)}{f(z)} dz. \tag{1}$$

Proof. Since $Z = \{\zeta_j\}$ is the set of all roots of $f(z)$ in D', there is a factorization

$$f(z) = (z - \zeta_1) \dots (z - \zeta_N) g(z)$$

where function $g(z)$ is holomorphic in D and does not have zeros in D'. Using the basic formula for the derivative of the product, we find:

$$\frac{f'(z)}{f(z)} = \frac{1}{z - \zeta_1} + \dots + \frac{1}{z - \zeta_N} + \frac{g'(z)}{g(z)}.$$

Function $g'(z)/g(z)$ is holomorphic in the whole D', so by Cauchy's theorem

$$\int_\gamma \frac{g'(z)}{g(z)} dz = 0.$$

Therefore:

$$\frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi \boldsymbol{i}} \sum_{j=1}^{N} \int_\gamma \frac{dz}{z - \zeta_j} = \sum_{j=1}^{N} n(\gamma, \zeta_j).$$

□

Remark 2. Argument principle implies that $I := \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f'(z)}{f(z)} dz$ is an integer. In particular, if we continuously *deform* γ and/or $f(z)$ then the value of I does not change as long as it is well-defined.

Given a parametrization $\gamma(t), t \in [a, b]$ ofγ, one can rewrite expression (1) as follows:

$$\frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi \boldsymbol{i}} \int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} dt = \frac{1}{2\pi \boldsymbol{i}} \int_a^b \frac{(f \circ \gamma)'(t)}{(f \circ \gamma)(t)} dt = \frac{1}{2\pi \boldsymbol{i}} \int_\Gamma \frac{dw}{w},$$

where $\Gamma = (f \circ \gamma)$ is a contour in the complex plane $\mathbb{C}$ with coordinate w. In other words.

$$\sum_j n(\gamma, \zeta_j) = n(\Gamma, 0). \tag{2}$$

A special, yet the most useful version of argument principle occurres if γ is a circle enclosing disk D'. In this case every $n(\gamma, \zeta_j) = 1$ for every zero inside D', and the integral (1) computes the number of zeros of $f(z)$ inside disk D'. In particular, equation (2) says that this number equals the winding number of Γ around the origin.

Example 3. If $P(z)$ is a polynomial of degree n, then for a large $R > 0$

$$\int_{|z|=R} \frac{P'(z)}{P(z)} dz = 2\pi \boldsymbol{i} n$$

Next theorem exploits the continuity of the integral $\int_\gamma f'(z)/f(z) dz$ in f to reduce computation of the number of zeros of f to a possibly simpler function.

Theorem 4 (Rouché's theorem)**.** *Suppose that functions $f(z)$ and $g(z)$ are holomorphic in a neighbourhood of a closed disk $\overline{D}$. If*

$$|f(z)| > |g(z)|$$

on the contour $\gamma = \partial D$, then f and $f + g$ have the same number of zeros inside D.

Proof. Consider a function $f_t(z) := f(z) + tg(z)$ and let n_t be the number of zeros of $f_t(z)$. Condition $|f| > |g|$ guarantees that $f + tg$ does not vanish on γ for $t \in [0,1]$, therefore

$$n_t = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f_t'(z)}{f_t(t)} dz.$$

The latter integral is well-defined and continuous in t for $t \in [0,1]$, and takes only integer vales, therefore it is constant. Hence $n_0 = n_1$. □

Remark 5. Rouché's theorem provide an alternative way to prove fundamental theorem of algebra. Given a polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ we can find R large enough such that $f(z) = z^n$ and $g(z) = P(z) - z^n$ satisfy the assumptions of Theorem 4. Hence we can conclude that $f(z) = z^n$ and $f(z) + g(z) = P(z)$ have the same number of zeros in a large enough disk.

Exercise 1. Determine the number of zeros of $f(z) = z^5 + 3z^2 - 1$ in $\{1 < |z| < 2\}$.

Open mapping theorem. Maximum principle

One of the important consequences of argument principle if open mapping theorem.

Definition 6. A continuous map $f : X \to Y$ is called *open*, if image of any open set $U \subset X$ is open in Y.

Exercise 2. Characterize open mappings $f : \mathbb{R} \to \mathbb{R}$.

Theorem 7 (Open mapping theorem)**.** *If $f(z)$ is holomorphic an non-constant in an open connected region, then $f(z)$ is open.*

Proof. Take arbitrary open subset U of the domain of f and let us prove that $f(U)$ is open.

Consider a point $z_0 \in U$ and define $w_0 := f(z_0)$. We will prove that a small neighbourhood of w_0 is contained in the image of U. Equivalently, we want to prove that function

$$g(z) := f(z) - w = \underbrace{(f(z) - w_0)}_{F(z)} + \underbrace{(w_0 - w)}_{G(z)}$$

has zeros for w in a small neighbourhood of w_0.

Choose $\delta > 0$ such that $B_\delta(z_0)$ is contained in U and $f(z) \neq w_0$ on $|z - z_0| = \delta$. Then select $\epsilon > 0$ such that $|f(z) - w_0| > \epsilon$ on $|z - z_0| = \delta$.

Now, as long as $|w - w_0| < \epsilon$, on $|z - z_0| = \delta$ we have $|F(z)| > \epsilon$, while $|G(z)| < \epsilon$, therefore, by Rouché's theorem function $g(z) = F(z) + G(z)$ has a zero inside $|z - z_0| < \delta$, since $F(z)$ has a zero. □

Remark 8. Carefully analyzing the proof of the above theorem, we can strengthen its statement. In fact, in a small neighbourhood of w_0 equation $f(z) = w$ has exactly n solutions (with multiplicities), where n is the order of zero of $f(z) - w_0$. In particular, if $n = 1$, then $f(z)$ is locally one-to-one.

A trivial, yet a very powerful corollary of the Open mapping theorem is the *Maximum modulus principle.*

Theorem 9 (Maximum modulus principle)**.** *If the modulus of a holomorphic function $f(z) : U \to \mathbb{C}$ attains its maximum at $z_0 \in U$, then $f(z)$ is constant.*

Proof. Since $f(z)$ is non-constant, by open mapping theorem, for any z_0 there exists $\epsilon > 0$ such that the open ball $B_\epsilon(f(z_0))$ is in the range of $f(z)$. Therefore $|f(z_0)|$ cannot be the maximum of $|f(z)|$ for $z \in U$. □

Remark 10. Alternatively, maximum modulus principle could be prove by using Cauchy's formula

$$f(z_0) = \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \int_0^{2\pi} f(z_0 + re^{i\alpha}) d\alpha.$$

The above identity holds for any $r > 0$ such that $\overline{B_r(z_0)} \subset U$, therefore the function $f(z)$ must be constant in $B_r(z_0)$, which by "rigidity" of holomorphic functions implies that $f(z)$ is constant in the connected component of U containing z_0.

One special case in which maximum modulus principle is often applied is characterized by the following theorem.

Theorem 11. *If $f(z)$ is defined and continuous on a closed bounded set E and analytic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.*

Proof. Since E is compact, the maximum of $|f(z)|$ is attained at some point $z_0 \in E$. If z_0 is a point in the interior of E, then by the maximum principle, $f(z)$ must be constant. Therefore either $f(z)$ is non-constant and its maximum is attained at the boundary of E, or it is constant, and its maximum is attained at every point of E. □