Lecture 11

General form of Cauchy's theorem & Notion of simple connectivity

In this lecture, we will give two definitions of simply connected regions. Later in the course we will prove that these two definitions are equivalent, while so far we deduce Cauchy's theorem in its general form from one of the definitions.

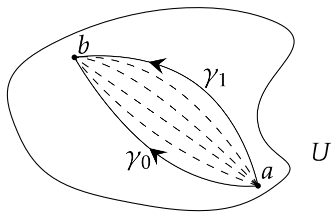
Definition 1. Let $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$ be two parametrized continuous paths between a and b. A (continuous) homotopy between γ_0 and γ_1 is a jointly continuous family of paths $\gamma_s(t)$, $s \in [0,1]$, i.e, a continuous map

$$\Gamma : [0,1] \times [0,1] \to \mathbb{C}, \qquad (s,t) \mapsto \gamma_s(t)$$

such that for any $s \in [0,1]$ we have $\gamma_s(0) = a$ and $\gamma_s(1) = b$.

Remark 2. Homotopy between paths induces an equivalence relation on the set of paths with fixed endpoints.

Definition 3. An open region $U \subset \mathbb{C}$ is *simply connected* if for any $a, b \in \mathbb{C}$ and any two paths between a and b there is a homotopy γ_s between these paths, lying entirely inside U.



Example 4. Any convex region U is simply connected. Indeed, if γ_0 and γ_1 are two paths, then we can use a standard *linear homotopy*:

$$\gamma_s(t) := (1-s)\gamma_0(t) + s\gamma_1(t).$$

Remark 5. If U is simply connected, then any loop γ is homotopic to a *constant* loop, i.e., a loop γ_{const} such that $\gamma_{\mathrm{const}}(t) = a$ for any t.

Theorem 6 (General from of Cauchy's theorem)**.** *If $f(z)$ is holomorphic in U, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

whenever paths γ_0 and γ_1 are homotopic in U.

Corollary 7. *If $f(z)$ is holomorphic in a simply connected region, then for any closed contour γ,*

$$\int_{\gamma} f(z)dz = 0.$$

Proof of corollary. Since region U is simply connected, a close contour γ is homotopic to a constant loop γ_c. Then by Theorem 6 we have

$$\int_{\gamma} f(z)dz = \int_{\gamma_c} f(z)dz$$

and the latter integral is clearly zero. □

Proof of the theorem. The proof will contain several steps.

Step 1. Let $K = \Gamma([0,1]\times[0,1])$ be the image the homotopy $\Gamma(s,t)$ between $\gamma_0(t)$ and $\gamma_1(t)$. Since $[0,1]\times[0,1]$ is compact, its continuous image K is also compact. As K is compact and $\mathbb{C} - \{U\}$ is closed, there exists $\epsilon > 0$ such that for every $z \in K$ and $w \in \mathbb{C} - \{U\}$

$$|z - w| > 2\epsilon,$$

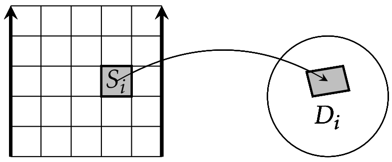
i.e., $B_{2\epsilon}(z)$ is contained in U. Fix such ϵ.

Step 2. Using uniform continuity of Γ on $[0,1]\times[0,1]$ we can find $\delta > 0$ such that

$$|\gamma_{s_1}(t_1) - \gamma_{s_2}(t_2)| < \epsilon,$$

whenever $|s_1 - s_2| + |t_1 - t_2| < \delta$. Fix this δ.

Step 3. Divide square $[0,1]\times[0,1]$ into N small squares with sides $< \delta$. By our choice of δ, image of every square S_i is contained in a disk D_i of radius 2ϵ such that $U \supset D_i \supset \Gamma(S_i)$. Since $f(z)$ is holomorphic in U, we also have that $f(z)$ is holomorphic in D_i.



By the usual Cauchy's theorem, we know that $f(z)$ has a primitive in D_i, therefore, the integral over the image of the boundary of S_i vanishes:

$$\int_{(\Gamma(\partial S_i))} f(z)dz = 0.$$

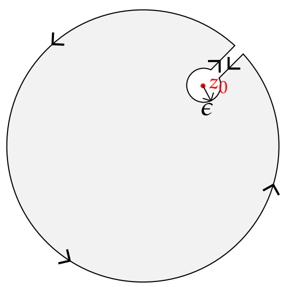
Adding up all such integrals over all N squares S_i with appropriate orientations, we conclude that

$$\int_{\gamma_0 - \gamma_1} f(z)dz = 0$$

Equivalently $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$. $\square$

Remark 8. While simply connectivity is a very intuitive notion, often it is hard to prove rigorously that a given region is simply connected. We will not be focusing on such proofs in the course and you can use Cauchy's theorem without formal explanation why the involved region is simply connected.

Example 9. Consider a function $f(z)$ holomorphic in D and $z_0 \in D$. Then one could prove Cauchy's integral formula by applying Cauchy's theorem to the function $\frac{f(z)}{z-z_0}$ in a region enclosed by a *keyhole* contour γ below.



If we denote the outer circle by γ_r and the inner circle by γ_ϵ, then letting the width of the "corridor" go to zero we conclude:

$$\int_{\gamma_r} \frac{f(z)}{z-z_0}dz = \int_{\gamma_\epsilon} \frac{f(z)}{z-z_0}dz.$$

Making $\epsilon \to 0$ it is easy to prove that the latter integral converges to $2\pi \boldsymbol{i} f(z_0)$:

$$\int_{\gamma_\epsilon} \left(\frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{o(z-z_0)}{z-z_0} \right) dz \to 2\pi \boldsymbol{i} f(z_0).$$

Another approach to simple connectivity in $\mathbb{C}$

The advantage of our first definition of simply connected sets $U \subset \mathbb{C}$ is its *universality*. It makes sense, as long as we can make sense of continuous maps. A drawback is that in our specific situation it is often hard to check simple connectivity formally. To this end we give the following definition specific to the regions in $\mathbb{C}$.

Definition 10. An open connected set $U \subset \mathbb{C}$ is *simply connected* if its complement in the extended complex plane $\hat{\mathbb{C}}$ is connected.

Example 11. According to this new definition, it is easy to see that disk D, plane $\mathbb{C}$, strip $\{0 < \operatorname{Im} z < 1\}$ are all simply connected regions, while annulus $\{1 < |z| < 2\}$ is not.

Example of a strip show that it is important to take complement in $\hat{\mathbb{C}}$ and not on $\mathbb{C}$.

Theorem 12. *An open set $U \subset \mathbb{C}$ is simply connected in the sense of Definition 10 if and only if for any closed curve $\gamma \subset U$ and any $z_0 \in \mathbb{C} - \{U\}$*

$$n(\gamma, z_0) = 0.$$

Proof. Let us first prove the necessary condition. Assume that U is simply connected and take $\gamma \subset U$. Then the complement $K = \hat{\mathbb{C}} - \{U\}$ is contained in one of the regions defined by γ. Since $\infty \in K$, we know that this must be the unbounded region defined by γ. But from the properties of winding number we know that

$$n(\gamma, z_0) = 0$$

whenever z_0 belongs to the unbounded region.

We prove the sufficient condition by a direct construction. Specifically, we will show that if region U is not simply connected, then one can construct a curve γ in U and find a point z_0 which does not belong to U such that $n(\gamma, z_0) \neq 0$. Let us assume that the complement of U in $\hat{\mathbb{C}}$ is a disjoint union of two closed sets $A \sqcup B$. If B is unbounded, A must be bounded.

Let $\delta > 0$ be the distance between A and B. Cover A with a grid of squares of size $\delta/10$. Let $S = \cup S_i$ be the union of squares which intersect A. We take

$$\gamma = \partial S.$$

Indeed, by our construction no point on ∂S can lie neither on A (because otherwise there will be another square intersecting A) nor on B (because B is "too far away").

Now take any $z_0 \in A$ which is not on the boundary of any of the squares. With appropriate choice of orientation on the boundaries of squares of S_i, we have $\gamma = \sum \partial S_i$. On the other hand z_0 is contained inside exactly one of the squares S_i. Hence $n(\partial S_i, z_0)$ is zero for all squares except for exactly one. $\square$

Complex logarithm in simply connected regions

Theorem 13. *If function $f(z)$ is holomorphic and nonzero in a simply connected open region U, then there exists a single-valued function $\operatorname{Log}_U(f(z))$ such that*

$$\exp(\operatorname{Log}_U(f(z))) = f(z).$$

Proof. Function $\frac{f'(z)}{f(z)}$ is holomorphic in U and by the general version of Cauchy's theorem all its integrals over closed contours γ vanish:

$$\int_\gamma \frac{f'(z)}{f(z)} = 0.$$

Therefore we can define its primitive $F(z)$:

$$F'(z) = \frac{f'(z)}{f(z)}.$$

Then

$$\frac{d}{dz}\left(f(z)e^{-F(z)}\right) = 0 \Longleftrightarrow f(z) = A\exp(F(z)), A \in \mathbb{C} - \{0\}.$$

It remains to fix any complex logarithm of a nonzero complex number A and set

$$\operatorname{Log}_U(f(z)) := F(z) + \operatorname{Log}(A).$$

$\square$

Remark 14. Under the assumptions of the above theorem definition of n-th roots also easily follows:

$$\sqrt[n]{f(z)} := \exp\left(\frac{1}{n}\operatorname{Log} f(z)\right)$$