Lecture 12

Automorphisms of basic complex regions

$\mathbb{D}$

We start the lecture with an important technical tool useful for studying self-maps of bounded domains.

Theorem 1 (Schwarz's lemma)**.** *If $f(z)$ is holomorphic in $\mathbb{D}$ and satisfies $|f(z)| \leqslant 1$, $f(0) = 0$ then*

$$|f(z)| \leqslant |z| \quad \textit{and} \quad |f'(0)| \leqslant 1. \tag{1}$$

If for some $z \neq 0$ one of the inequalities of (1) turns into the equality, then $f(z) = az$ for some constant a with $|a| = 1$.

Proof. Since $f(z)$ has zero at $z = 0$, the quotient $f_1(z) := \frac{f(z)}{z}$ extends to a well-defined holomorphic function in $\mathbb{D}$.

On a circle $|z| = r < 1$ we have that $|f_1(z)| \leqslant 1/r$. Therefore, by maximum principle the same is true in the disk $\{|z| \leqslant r\}$. Since $r \in (0,1)$ is arbitrary, we conclude that $|f_1(z)| < 1$, which is equivalent to the inequality statement of the theorem.

If equality occurs at a point z_0, then $|f_1(z)|$ attains maximum at an interior point, hence $f_1(z)$ must be a constant. □

Theorem 2 (Schwarz-Pick theorem)**.** *Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic. Then for all $z_1, z_2 \in \mathbb{D}$*

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leqslant \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|$$ [1]

Schwarz lemma allows to classify all automorphisms of the unit disk. First, note that for any $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and any $b \in \mathbb{D}$ the map

$$F_{\alpha,b}(z) := \alpha \frac{z - b}{1 - z\bar{b}} \tag{2}$$

is an automorphism of $\mathbb{D}$, and the set of such maps (2) form a group.

Theorem 3. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be an holomorphic bijective map. Then f must be of the form $F_{\alpha,b}$ for some b and α.*

Proof. Consider a map $g(z) := (F_{1,f(0)} \circ f)(z)$. This map is an automorphism of $\mathbb{D}$ such that $g(0) = 0$. Clearly, either $g'(0)$ or $(g^{-1})'(0) = 1/g'(0)$ must have modulus $\geqslant 1$. Therefore by Schwarz lemma we have that $g(z) = \alpha z$ for some α. Hence

$$f(z) = F^{-1}_{1,f(0)}(\alpha z)$$

which is also of the form (2). □

Proof. Compose $f(z)$ with a fractional automorphisms of $\mathbb{D}$

$$g(z) := (F_2 \circ f \circ F_1)(z)$$

such that $F_1\colon 0 \mapsto z_1$ and $F_2\colon f(z_1) \mapsto 0$. Then g satisfies the usual Schwarz lemma

$$|g(z)| \leqslant |z|.$$

which implies the stated inequality with $z_2 := F_1^{-1}(z)$. □

[1] In other words, $f(z)$ can not increase hyperbolic distance in $\mathbb{D}$.

$\mathbb{C}$

If $f : \mathbb{C} \to \mathbb{C}$ is an holomorphic bijective map, then f must by of the form

$$F_{a,b}(z) = az + b$$

by the problem form homework. Let us record its proof for completeness.

Proof. Function $f(z)$ has an isolated singularity at ∞. We claim that it must be a pole of order 1.

First, we rule out essential singularity. If $\infty \in \widehat{\mathbb{C}}$ is an essential singularity, then f maps the complement of $\overline{D}$ to an open everywhere dense set Z. Such set Z must intersect the open set $f(\mathbb{D})$, which contradicts injectivity of $f(z)$.

Clearly $f(z)$ can't have a removable singularity at ∞, since in this case $f(z)$ would be entire and bounded, and Liouville theorem implies that such $f(z)$ is constant.

Hence $f(z)$ has a pole at ∞. But we know that the function which has only poles as its singularities must be a rational function $P(z)/Q(z)$. Rational function is injective if and only if it is a polynomial of degree 1. □

$\widehat{\mathbb{C}}$

We already mentioned that any function of the form

$$F(z) = \frac{az + b}{cz + d} \tag{3}$$

with $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ is an automorphism of $\widehat{\mathbb{C}}$. We claim that there no other automorphisms of $\widehat{\mathbb{C}}$. Indeed, if $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is such a map, then we can compose it with a map F of the form (3), which takes $f(\infty)$ to ∞. Then $F \circ f$ is an automorphism of $\mathbb{C}$, and those, as was already described, are of also of the form (3).