Lecture 13

Calculus of residues

Residues

Assume that function $f(z)$ is holomorphic in a region $U-\{z_0\}$ and has a singularity at z_0. As we know, in under this assumption, Cauchy's theorem is not necessarily valid, in particular, for a circle $C \subset U$ centered at z_0 the integral

$$\int_C f(z)dz$$

might be non-zero.

To capture the failure of Cauchy's theorem quantitatively, we introduce a notion of *residue* of $f(z)$ at z_0.

Definition 1. A *residue* of $f(z)$ at z_0 is

$$\mathrm{res}_{z_0} f(z) := \frac{1}{2\pi \boldsymbol{i}} \int_C f(z)dz$$

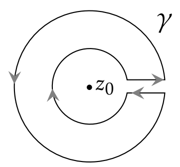
where $C \subset U$ is a circle centered at z_0.

Lemma 2. *The residue* $\mathrm{res}_{z_0} f(z)$ *does not depend on the choice of circle* C *centered at* z_0.

Proof. Let C_r and C_R be two circles centered at z_0 of radii r and R respectively. We claim that

$$\int_{C_r} f(z)dz = \int_{C_R} f(z)dz.$$

Indeed consider a keyhole-like contour γ joining two circles of radii r and R.



By the general form of Cauchy's theorem we have $\int_\gamma f(z)dz = 0$. Hence, letting the *corridor* width to go to zero, we find

$$0 = \int_\gamma f(z)dz = \int_{C_R} f(z)dz - \int_{C_r} f(z)dz.$$

□

Remark 3. Clearly, if $\rho = \mathrm{res}_{z_0} dz$, then

$$\int_C \left(f(z) - \frac{\rho}{z-z_0} \right) dz = 0 \tag{1}$$

for any circle C centered at z_0. Moreover, number $\rho \in \mathbb{C}$ such that (1) hold is unique.

Exercise 1. Any closed curve γ in an annulus $\{r < |z| < R\}$ is homotopic to a curve $C_n(t) = r' e^{2\pi \boldsymbol{i} n t}$, where $r' \in (r, R)$, $t \in [0,1]$ and $n = n(\gamma, 0)$.

In other words, $C_n(t)$ winds around 0 exactly $n = n(\gamma, 0)$ times with a constant angular speed.

Lemma 4. *For a ball* $B_\epsilon(z_0) \subset U$, *function* $f(z) - \frac{\rho}{z-z_0}$ *has a single-valued antiderivative in* $B_\epsilon(z_0) - \{z_0\}$.

Proof. By the exercise, any curve γ is homotopic to some curve C_n, and by (1) we have

$$\int_\gamma f(z)dz = \int_{C_n} f(z)dz = n \int_C f(z)dz = 0.$$

Since $\int_\gamma f(z)dz = 0$ for any curve $\gamma \subset B_\epsilon(z_0) - \{z_0\}$, function $f(z)$ has an antiderivative in this region. □

Residue at z_0 is hard to compute in general if function $f(z)$ has an essential singularity at z_0. The situation is entirely different if z_0 is a pole. As we know, in this case, we can isolate the principle part of the pole:

$$f(z) = \underbrace{\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{z-z_0}} + \varphi(z) \tag{2}$$

where $\varphi(z)$ is a holomorphic function in neighborhood of z_0. Let C be a circle centered at z_0.

The contour integral $\int_C$ for all terms of (2) vanishes except for the integral

$$\int_C \frac{a_{-1}}{z-z_0} dz = 2\pi \boldsymbol{i} a_{-1}. \tag{3}$$

Equation (3) gives a particularly simple formula for the residue of a function at its pole:

$$\boxed{\operatorname{res}_{z_0} f(z) = a_{-1}}$$

where a_{-1} is the coefficient in front of $(z-z_0)^{-1}$ term in the pole's principle part.

Remark 5. If function $f(z)$ has a *simple* pole at z_0 (i.e., of order 1), then a_{-1} can be recovered as

$$a_{-1} = \lim_{z \to z_0} (z-z_0) f(z).$$

In general, if the order of the pole is $n \geqslant 1$, one has the following formula

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} ((z-z_0)^n f(z))^{(n-1)}$$

which follows immediately from (2).

Example 6. Function $f(z) = \frac{e^z}{z^n}$ has a pole of order n at $z = 0$. To compute its residue, we use the Taylor's expansion of e^z:

$$e^z = \sum_{z=1}^{n} \frac{z^n}{n!}$$

and find that the principle part of $f(z)$ to be

$$\sum_{i=0}^{n-1} \frac{1}{i! z^{n-i}}$$

so $\operatorname{res}_0 f(z) = a_{-1} = \frac{1}{(n-1)!}$

Residue theorem

Assume that curve γ bounds an open region U[1]. Consider function $f(z)$ which is holomorphic in $U - \{z_1, \ldots z_k\}$, continuous in $\overline{U} - \{z_1, \ldots, z_k\}$ and has isolated singularities at z_i.

Theorem 7. *For function $f(z)$ as above we have*

$$\int_\gamma f(z) dz = 2\pi \boldsymbol{i} \sum_{i=1}^{k} \operatorname{res}_{z_i} f(z). \tag{4}$$

Proof. Enclose each z_i in a small circle C_i and connected all C_i's with γ with non-intersecting arcs γ_i. Then we can arrange C_i's (with opposite orientations), γ and γ_i's in a keyhole-like contour Γ. Contour Γ bounds a simply region $U - \cup_i \gamma_i$.

Hence

$$\int_\Gamma f(z) dz = 0$$

On other hand,

$$\int_\Gamma f(z) dz = \int_\gamma f(z) dz - \sum_i \int_{C_i} f(z) dz = \int_\gamma f(z) dz - 2\pi \boldsymbol{i} \sum_i \operatorname{res}_{z_i} f(z)$$

which concludes the proof. □

[1] As always, we assume that curve γ is *positively oriented*, which means that region U stays on the *left* as we follow γ

Identity (4) can be used in two directions: first is a powerful tool for the evaluation of definite integrals, as we will see later. On other hand, it could be used to estimate residues and locate the poles of a given function.

One can also prove the following stronger version of Residue theorem, but we will not need it in our course.

Theorem 8. *Let $\gamma \subset U$ be a contractible loop, such that $z_i \notin \gamma$. Then for any function $f(z)$ as above we have*

$$\int_\gamma f(z)dz = \sum_i n(\gamma, z_i)\mathrm{res}_{z_i} f(z).$$

In our previous setting γ was the boundary of a simply connected region and all $n(\gamma, z_i) = 1$.

Evaluation of definite integrals

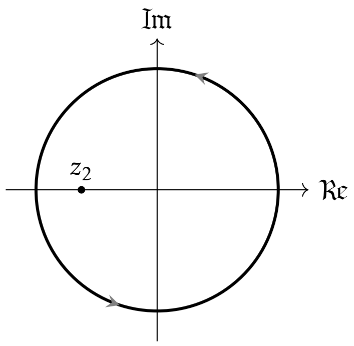
In this section, through a series of examples we demonstrate how to use residues formula to evaluate various definite integrals.

Example 9. Consider integral

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta}$$

where $a > 1$ is a constant. If we let $z = e^{\boldsymbol{i}\theta}$, then $dz = \boldsymbol{i}e^{\boldsymbol{i}\theta}d\theta$ and $\cos\theta = \frac{1}{2}(z + z^{-1})$ so the integral can be rewritten as

$$\int_{|z|=1} \frac{1}{a + \frac{1}{2}(z + z^{-1})}\frac{dz}{\boldsymbol{i}z} = -2\boldsymbol{i}\int_{|z|=1} \frac{dz}{z^2 + 2az + 1}.$$



The integrand $f(z) = 1/(z^2 + 2az + 1)$ has poles at $z_1 = -a - \sqrt{a^2 - 1}$ and $z_2 = -a + \sqrt{a^2 - 1}$. Clearly $|z_1| > 1$ and $|z_2| < 1$, hence function $f(z)$ has only one pole at z_2 in the unit disk $\mathbb{D}$, and

$$\mathrm{res}_{z_2} \frac{1}{z^2 + 2az + 1} = \lim_{z \to z_2}(z - z_2)\frac{1}{(z - z_1)(z - z_2)} = \frac{1}{z_2 - z_1} = \frac{1}{2\sqrt{a^2 - 1}}$$

Therefore

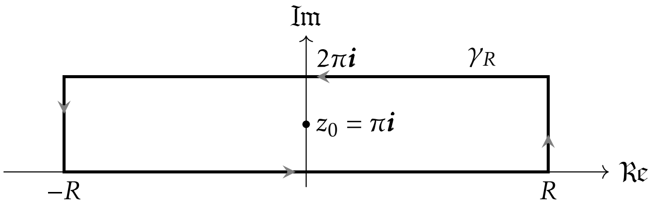
$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = 2\pi\boldsymbol{i} \cdot (-2\boldsymbol{i}) \cdot \frac{1}{\sqrt{a^2 - 1}} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

A very common application of residue theorem occurs when we want to compute a definite integral of the form $\int_{-\infty}^{+\infty} f(x)dx$. In the case we first must choose a closed contour γ such that the integral $\int_\gamma f(z)dz$ can be estimated in terms of $\int_{-\infty}^{\infty} f(x)dx$.

Example 10.

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x}dx, \quad 0 < a < 1.$$

Let $f(z) = \frac{e^{az}}{1+e^z}$. Consider the contour γ_R consisting of a rectangle with vertices at $-R, R, R + 2\pi\boldsymbol{i}, -R + 2\pi\boldsymbol{i}$.



Inside this contour $f(z)$ has only one pole at $z_0 = \pi \boldsymbol{i}$, and residue at this point is

$$\lim_{z\to\pi \boldsymbol{i}}(z-\pi\boldsymbol{i})f(z) = e^{a\pi\boldsymbol{i}}\lim_{z\to\pi\boldsymbol{i}}\frac{z-\pi\boldsymbol{i}}{e^z - e^{\pi\boldsymbol{i}}}$$

The last limit is the inverse of

$$\lim_{z\to\pi\boldsymbol{i}}\frac{e^z - e^{\pi\boldsymbol{i}}}{z-\pi\boldsymbol{i}} = (e^z)\Big|'_{z=\pi\boldsymbol{i}} = e^{\pi\boldsymbol{i}} = -1,$$

hence

$$\mathrm{res}_{\pi\boldsymbol{i}} f(z) = -e^{a\pi\boldsymbol{i}}.$$

Now we investigate the integrals of $f(z)$ over the sides of rectangle. If we denote by I_R the integral over the lower side of the rectangle:

$$I_R = \int_{-R}^{R} f(z)dz,$$

then $I_R \to I$ as $R\to\infty$, where I is the integral of interest. The integral over the top side of the rectangle is then

$$-e^{2\pi\boldsymbol{i}a}I_R$$

(with minus sign because of the opposite orientation). Finally integrals over the vertical sides can be bounded as

$$\int_0^{\pm R+2\pi\boldsymbol{i}} f(z)dz \leqslant \int_0^{2\pi}\left|\frac{e^{a(R+\boldsymbol{i}t)}}{1+e^{R+\boldsymbol{i}t}}\right|dt \leqslant Ce^{(a-1)R}.$$

Since $a<1$ these integrals go to zero as $R\to\infty$. Collecting everything together we find:

$$I - Ie^{2\pi\boldsymbol{i}a} = 2\pi\boldsymbol{i}\,\mathrm{res}_{\pi\boldsymbol{i}} f(z) = -2\pi\boldsymbol{i}e^{a\pi\boldsymbol{i}}.$$

and

$$I = -2\pi\boldsymbol{i}\frac{e^{a\pi\boldsymbol{i}}}{1-e^{2a\pi\boldsymbol{i}}} = \pi\frac{2\boldsymbol{i}}{e^{a\pi\boldsymbol{i}} - e^{-a\pi\boldsymbol{i}}} = \frac{\pi}{\sin \pi a}.$$