# Lecture 14

## Laurent series

Assume that function $f(z)$ is holomorphic and in a complement of a closed disk

$$\mathbb{C} - \overline{B_R}(0) = \{|z| > R\}$$

and has a removable singularity at $\infty$. In this case function $g(z) := f(1/z)$ is holomorphic in a disk $B_{1/R}(0)$, and therefore can be represented by a convergent Taylor's series:

$$g(z) := f(1/z) = \sum_{i=0}^{\infty} a_i z^i.$$

or equivalently

$$f(z) = \sum_{i=-\infty}^{i=0} a_{-i} z^i. \tag{1}$$

Expression of the form (1) is a particular example of a *Laurent series*. As we will now show similar representation can be found for any function holomorphic in an annulus $\{r < |z| < R\}$.

**Theorem 1.** *Let $f(z)$ be a function holomorphic in $\{r < |z| < R\}$ then there exists a sequence of complex numbers $\{a_i\}_{i\in\mathbb{Z}}$ such that*

- *power series $\sum_{i=0}^{\infty} a_i z^i$ absolutely converges in $\{|z| < R\}$*
- *power series $\sum_{i=-\infty}^{0} a_i z^i$ absolutely converges in $\{|z| > r\}$*
- *the sum of the above power series represents $f(z)$:*

$$f(z) = \sum_{-\infty}^{+\infty} a_i z^i.$$

**Remark 2.** As a consequence of the above theorem, any function $f(z)$ holomorphic in an annulus can be represented as a sum of two function $f_1(z)$ and $f_2(z)$ holomorphic in $\{|z| < R\}$ and $\{|z| > r\}$ respectively.

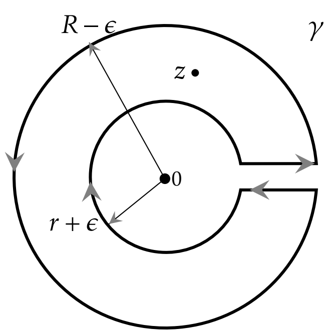
*Proof.* We start the proof with a lemma

**Lemma 3.** *Function $f(z)$ as above can be represented as*

$$f(z) = \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=R-\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta$$

*where $\epsilon > 0$ is chosen in such a way that $r + \epsilon < |z| < R - \epsilon$.*

*Proof of the lemma.* Consider a keyhole contour $\gamma$ with radii $R - \epsilon$ and $r + \epsilon$.



Then, by general version of Cauchy's theorem (or by residue theorem)

$$f(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=R-\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta$$

This proves the lemma. □

Once the lemma is prove we can proceed exactly the same way as with the Taylor's series. Specifically,

Function $f_1(z) = \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=R-\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta$ is a holomorphic function in the disk $B_{R-\epsilon}(0)$ and can be represented by a convergent power series in this disk:

$$f_1(z) = \sum_{i=0}^{\infty} a_i z^i.$$

Similarly for a function $f_2(z) = \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta$, using the fact that $|\zeta/z| < 1$ we can rewrite the defining identity as

$$f_2(z) = \frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=r+\epsilon} \frac{f(\zeta)}{\zeta - z} d\zeta = -\frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=r+\epsilon} f(\zeta) \sum_{i=0}^{\infty} \frac{\zeta^i}{z^{i+1}} d\zeta = \sum_{i=0}^{\infty} \frac{b_i}{z^{i+1}},$$

where $b_i = -\frac{1}{2\pi \boldsymbol{i}} \int_{|\zeta|=r+\epsilon} f(\zeta) \zeta^i d\zeta$. (As with Taylor's series, we can interchange integration and summation, since the power series is absolutely convergent.) □

**Remark 4.** Special case of the above theorem is $r = 0$. In this case, function $f(z)$ has an isolated singularity at $z = 0$. This singularity if removable all $a_{-k}, k \in \mathbb{N}$ vanish, and is a pole $a_{-k}, k > C$ vanish.

The proof of the theorem, in particular, shows that

$$\mathrm{res}_0 f(z) = a_{-1}$$

even for an essential singularity.

# General form of the argument principle

We have proved that if $f(z)$ is holomorphic in an open disk $B_R(z_0)$ and $\gamma \subset B_R(z_0)$ is a closed curve inside the disk then

$$\int \frac{f'(z)}{f(z)} dz = 2\pi \boldsymbol{i} \sum n(\gamma, \zeta_i),$$

where the sum is taken over all zeros of $f(z)$ in $B_R(z_0)$ and all zeros are counted with their multiplicities.

With the residue theorem we can prove an improved version of this argument principle.

**Theorem 5.** *Assume that curve $\gamma$ bounds a simply connected region $U$, and function $f(z)$ is meromorphic in a neighbourhood of $U$. If $\gamma$ does not contain zeros or poles of $U$ then*

$$\int \frac{f'(z)}{f(z)} dz = 2\pi \boldsymbol{i}(\#zeros - \#poles),$$

*where #zeros and #poles are the numbers of zeros and poles of $f(z)$ in $U$ counted with multiplicities.*

*Proof.* If function $f(z)$ has a pole of order $k$ at $z_0$, then we can factor $f(z)$ as $f(z) = (z - z_0)^{-k} g(z)$, $g(z_0) \neq 0$ and compute

$$\frac{f'(z)}{f(z)} = \frac{-k}{z - z_0} + \frac{g'(z)}{g(z)},$$

i.e., $f'/f$ has a pole of order 1 with residue $-k$. Similarly, if $z_0$ is a zero of order $k$, then $f'/f$ has a pole of order 1 with residue $k$.

Applying residue theorem to function $f'/f$ we conclude:

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi \boldsymbol{i} \left( \sum_{z_0 \in \{\mathrm{zeros}\}} \mathrm{ord}(z_0) - \sum_{z_0 \in \{\mathrm{poles}\}} \mathrm{ord}(z_0) \right)$$

which is exactly the stated identity. □

## Harmonic functions

As we have already seen, the theory of holomorphic function is parallel to the theory of *harmonic* functions on $\mathbb{R}^2$. Today we will further investigate the links between these two topics. Throughout this topic we identify a point $(x,y) \in \mathbb{R}^2$ with the complex number $z = x + \boldsymbol{i}y \in \mathbb{C}$. In particular we will use interchangeably $u(x,y)$ and $u(z)$.

**Definition 6.** Twice differentiable function $u\colon (x,y) \in U \to \mathbb{R}$ is harmonic if

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \forall (x,y) \in U.$$

From Cauchy-Riemann equations we know that if function $f(z)$, $z = x + \boldsymbol{i}y$ is holomorphic in $U$, then $u := \mathfrak{Re} f$ and $v := \mathfrak{Im} f$ are harmonic in $U$. Function $v\colon (x,y) \in U \to \mathbb{R}$ is called the *conjugate* harmonic function of $u(x,y)$.

**Question.** *Given a harmonic function $u(x,y)$ in an open region $U$, does there exist a conjugate function $v(x,y)$?*

**Example 7.** Function $u(x,y) = x^2 - y^2$ is harmonic in $\mathbb{R}^2$ with conjugate $v(x,y) = 2xy$. Indeed $f(z) = (x^2 - y^2) + \boldsymbol{i}(2xy) = z^2$ is clearly holomorphic.

**Example 8.** Function $\log(x^2 + y^2)$ is harmonic in $\mathbb{R}^2 - \{(0,0)\}$ but does not have a conjugate function in this region. Of course, the reason is that there is no single-valued logarithm function $\log(z)$ in $\mathbb{C}^2 - \{0\}$.

**Proposition 9.** *If function $u(x,y)$ is harmonic in an open set $U$, then*

$$f(z) := \frac{\partial u}{\partial x} - \boldsymbol{i}\frac{\partial u}{\partial y}$$

*is holomorphic in $U$.*

*Proof.* $\mathfrak{Re}(f)$ and $\mathfrak{Im}(f)$ satisfy Cauchy-Riemann equations. □

The following theorem provides a very general sufficient condition for the existence of a conjugate harmonic functions.

**Theorem 10.** *Let $u(x,y)$ be a harmonic function in an open, connected,* ***simply connected*** *region $U$. Then $u(x,y)$ admits a conjugate function $v(x,y)$. Moreover, $v(x,y)$ is unique up to an additive constant.*

*Proof.* Function

$$f(z) := \frac{\partial u}{\partial x} - \boldsymbol{i}\frac{\partial u}{\partial y}$$

is holomorphic in a simply-connected $U$, therefore $f(z)$ admits a primitive $F(z)$:

$$F'(z) = f(z). \tag{2}$$

Claim: $\mathfrak{Re}(F) = u + const$. Indeed equation (2) implies that

$$\frac{\partial \mathfrak{Re} F}{\partial x} = \frac{\partial u}{\partial x}, \quad \frac{\partial \mathfrak{Re} F}{\partial y} = \frac{\partial u}{\partial y}.$$

So function $(\mathfrak{Re} F - u)$ has vanishing partial derivatives in $U$. Since $U$ is connected, $\mathfrak{Re} F$ and $u$ must differ by a real constant.

Therefore $v(x,y) := (\mathfrak{Im} F)(x + \boldsymbol{i}y)$ is a conjugate harmonic function. If $v_0(x,y)$ is another conjugate function, then $(v - v_0)$ has zero partial derivatives and also must by a constant in a connected region. □

**Remark 11.** The above theorem has many important corollaries. Being holomorphic, function $F(z)$ is infinitely differentiable and can be represented by a convergent power series in $z$, therefore $u(x,y) = (\mathfrak{Re} F)(x + \boldsymbol{i}y)$ is infinitely $\mathbb{R}$-differentiable and can be represented by a convergent power series in $x$ and $y$. Thus any harmonic function is analytic.

Now, once we have established a precise correspondence between holomorphic and harmonic functions, our next goal will be to translate results of complex analysis into statements about harmonic functions.