Lecture 15

Mean-value theorem

Theorem 1 (Mean-value theorem)**.** *Suppose function $u\colon \overline{B_R(z_0)} \to \mathbb{R}$ is harmonic in an open disk $B_R(z_0)$ and continuous in its closure $\overline{B_R(z_0)}$. Then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{\boldsymbol{i}\theta})d\theta. \tag{1}$$

Proof. Since function u is harmonic in a simply connected region $B_R(z_0)$, we can find a holomorphic function $f\colon B_R(z_0) \to \mathbb{C}$ such that $\mathfrak{Re}(f) = u$.

By Cauchy's theorem, for any $r \in (0, R)$ we have

$$f(z_0) = \frac{1}{2\pi\boldsymbol{i}}\int_{|z-z_0|=r} \frac{f(\zeta)}{\zeta - z_0}d\zeta. \tag{2}$$

Now, choose parametrization $\gamma(\theta)$ of $\{|z - z_0| = r\}$:

$$\gamma(\theta) = z_0 + re^{i\theta}, \quad \theta \in [0, 2\pi]$$

and rewrite (2) substituting $\zeta = z_0 + re^{i\theta}$ and $d\zeta = \boldsymbol{i}re^{\boldsymbol{i}\theta}d\theta$ (equivalently $d\theta = \frac{d\zeta}{\boldsymbol{i}(\zeta - z_0)}$):

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

Considering the real part of the above identity and letting $r \to R$ we prove the theorem. □

Remark 2. In other words, the value of a harmonic function $u(z)\colon U \to \mathbb{R}$, at any point in $z_0 \in U$, equals the *average* value of $u(z)$ on (any) circle centered at z_0.

This reformulation of the mean-value theorem agrees with the physical interpretation of harmonic functions, as steady heat distributions.

Corollary 3 (Maximum principle)**.** *If harmonic function $u\colon U \to \mathbb{R}$ achieves its maximum(minimum) at an interior point $z_0 \in U$, then u is locally constant.*

Proof. Assume $u(z)$ achieves local maximum at $z_0 \in U$. Since $z_0 \in U$ is an interior point, we can find a small disk $\overline{B_R}(z_0) \subset U$. Applying the mean-value theorem to $u(z)$ and $B_r(z_0) \subset B_R(z_0)$, we find

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{\boldsymbol{i}\theta})d\theta.$$

Since for all $\theta \in [0, 2\pi]$ we must have $u(z_0 + re^{\boldsymbol{i}\theta}) \leqslant u(z_0)$, the above identity can only hold if

$$u(z_0 + re^{\boldsymbol{i}\theta}) = u(z_0) \text{ for all } \theta.$$

Since $r \in (0, R)$ is arbitrary, we conclude that $u(z)$ is locally constant. □

Corollary 4. *If u_1 and u_2 are two continuous functions on a closed bounded set $\overline{U}$ which are harmonic in the interior U of $\overline{U}$ and such that $u_1 = u_2$ on the boundary of $\overline{U}$, then $u_1 = u_2$ in U.*

In other words, harmonic functions as above are uniquely determined by their values on the boundary.

Poisson formula in a disk

Suppose function $u\colon \overline{\mathbb{D}} \to \mathbb{R}$ is harmonic in an open unit disk $\mathbb{D}$ and continuous in its closure $\overline{\mathbb{D}}$. Formula (1) allows to recover the value $u(0)$ from the values of u on $\partial\mathbb{D}$. On the other hand, Corollary 4 implies that values $u(w)$ at other points $w \in \mathbb{D}$ must be also uniquely determined by the values of u on $\partial\mathbb{D}$. The following theorem gives a precise formula for $u(w)$ at any fixed $w \in \mathbb{D}$ in terms of $u(z), z \in \partial\mathbb{D}$.

Theorem 5 (Poisson formula in $\mathbb{D}$). *Let $u\colon \overline{\mathbb{D}} \to \mathbb{R}$ be as above. Then for $w \in \mathbb{D}$ we have*

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\theta}-w|^2} u(e^{i\theta})d\theta. \tag{3}$$

Remark 6. If $w = 0$, the above formula recovers mean-value theorem (1) in $\mathbb{D}$.

Proof. The idea is to pre-compose function $u(z)$ with a holomorphic mapping of the disk

$$f\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}, \qquad f\colon \zeta \mapsto \frac{\zeta + w}{1+\overline{w}\zeta}$$

which sends $0 \in \mathbb{D}$ to w.

Then, function $U(\zeta) := u(f(\zeta))$ is harmonic as a composition of a holomorphic mapping and harmonic function[1]. In particular, $U(\zeta)$ satisfies mean value theorem:

$$u(w) = U(0) = \frac{1}{2\pi}\int_0^{2\pi} U(e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{e^{i\theta}+w}{1+\overline{w}e^{i\theta}}\right)d\theta \tag{4}$$

Since $f\colon \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ maps bijectively the boundary of disk on itself, we know that

$$\frac{e^{i\theta}+w}{1+\overline{w}e^{i\theta}} = e^{i\alpha},$$

or equivalently

$$e^{i\theta} = \frac{w-e^{i\alpha}}{\overline{w}e^{i\alpha}-1}.$$

for a function $\alpha\colon \theta \in [0,2\pi] \to [0,2\pi]$. Taking differential of both sides of the above identity, we find:

$$d\theta = -\frac{e^{i\alpha}d\alpha}{w-e^{i\alpha}} - \frac{\overline{w}e^{i\alpha}d\alpha}{\overline{w}e^{i\alpha}-1} = d\alpha\left(\frac{e^{i\alpha}}{e^{i\alpha}-w} + \frac{\overline{w}}{e^{-i\alpha}-\overline{w}}\right) = \frac{1-|w|^2}{|e^{i\alpha}-w|^2}d\alpha.$$

Substituting $d\theta$ back into (4), we find

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\alpha}-w|^2} u(e^{i\alpha})d\alpha.$$

as required. □

Remark 7. Function

$$K_w(z) := \frac{1-|w|^2}{|z-w|^2} = \mathfrak{Re}\left(\frac{z+w}{z-w}\right), \quad |z| = 1$$

is called *Poisson kernel* in the unit disk.

Of course, there is an analogue of Poisson formula in any disk $B_R(z_0)$.

With the second expression for the Poisson kernel, we can rewrite Poisson formula (3) as

$$u(w) = \mathfrak{Re}\left(\frac{1}{2\pi i}\int_{|z|=1} \frac{z+w}{z-w}\frac{u(z)}{z}dz\right).$$

The function in parenthesis is a holomorphic function in $w \in \mathbb{D}$. In particular, its imaginary part provides a conjugate harmonic function for u. Moreover, given continuous $u\colon \partial\mathbb{D} \to \mathbb{R}$, we can define $u(w)$ for $w \in \mathbb{D}$ by the above integral formula and the result will be a harmonic function.

Question. *Given the above observation, it is natural to ask if such $u(w)$ defined by the Poisson formula (3) in the interior of $\mathbb{D}$ is continuous in $\overline{\mathbb{D}}$. In other words, is it true that for $z \in \partial\mathbb{D}$ we have $\lim_{w\to z} u(w) = u(z)$?*

This question will be answered in the next section.

[1] This is Problem 2 from the homework.

Schwarz's theorem

Theorem 8. *Given a piecewise continuous function $u(e^{i\theta})$, $\theta \in [0, 2\pi]$, the Poisson integral*

$$P_u(w) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|w|^2}{|e^{i\theta}-w|^2} u(e^{i\theta})d\theta$$

is harmonic for $|w| < 1$ and $\lim_{z\to e^{i\theta_0}} P_u(z) = u(e^{i\theta_0})$ if u is continuous at $e^{i\theta_0}$.

Proof. We already know that $P_u(w)$ is harmonic in $\mathbb{D}$ as a real part of a holomorphic function.

Let us list properties of P_u.

1. $P_{u_1+u_2} = P_{u_1} + P_{u_2}$;
2. $P_{cu} = cP_u$ for a constant $c \in \mathbb{R}$;
3. $P_{u_1} \geqslant P_{u_2}$ on $\mathbb{D}$ as long as $u_1 \geqslant u_2$ on $\partial\mathbb{D}$;
4. $P_{\mathbf{1}} = \mathbf{1}$.

Properties 1 and 2 are obvious; property 3 follows fomr the positivity of Poisson kernel: $K_w(z) > 0$; property 4 follows from the Poisson formula applied to a harmonic function $u(z) = \mathbf{1}$. Note that 2. and 4. imply $P_c = c$ for any constant $c \in \mathbb{R}$.

Now let $e^{i\theta_0}$ be a point of continuity of u. Considering $u - u(e^{i\theta_0})$ instead of u we may assume that $u(e^{i\theta_0}) = 0$.

Since u is continuous at $e^{i\theta_0}$, for any $\epsilon > 0$ we can find a small open arc $C_1 \subset \partial\mathbb{D}$ containing $e^{i\theta_0}$ such that $|u(e^{i\theta})| < \epsilon$ for $\theta \in C_1$. Let C_2 be the complementary arc.

We can split $u(z) = u_1(z) + u_2(z)$, where[2] $u_1(z) := \chi(z \in C_1)\cdot u(z)$ is supported on C_1 and $u_2(z) := \chi(z \in C_2)\cdot u(z)$ is supported on C_2.

We have $|u_1| < \epsilon$ on $\partial\mathbb{D}$, hence by properties of P_u, we conclude $|P_{u_1}| < \epsilon$ in $\mathbb{D}$.

On the other hand P_{u_2} can be expressed as

$$P_{u_2}(w) = \frac{1}{2\pi}\int_{C_2} \frac{1-|w|^2}{|e^{i\theta}-w|^2} u(e^{i\theta})d\theta$$

This integral yields a well-defined harmonic function for all $w \notin C_2$. In particular, P_{u_2} is continuous on C_1. Moreover, clearly $P_{u_2}(w) = 0$ for $w \in C_1$, since the Poisson kernel vanishes on C_2.

Therefore, in some δ-neighbourhood of $e^{i\theta_0}$ we have $|P_{u_2}| < \epsilon$. We conclude that for $w \in B_\delta(e^{i\theta_0}) \cap \mathbb{D}$

$$|P_u(w)| = |P_{u_1}(w) + P_{u_2}(w)| \leqslant |P_{u_1}(w)| + |P_{u_2}(w)| \leqslant 2\epsilon.$$

Since ϵ is arbitrary, $P_u(w) \to 0$ as $w \in e^{i\theta_0}$. □

Schwarz's theorem ensures that given a continuous function on a boundary of a unit circle $\partial\mathbb{D}$, we can extend it to a continuous function, harmonic in $\mathbb{D}$.

Reflection principle

There is a common situation when a given harmonic (holomorphic) function $u(z)$ ($f(z)$) in a domain U can be explicitly extended to a harmonic (holomorphic) function in a larger domain.

Throughout this section, U is an open connected set which is symmetric with respect to x-axis, and let $U_+ := U \cap \{\mathrm{Im}(z) > 0\}$, $U_- := U \cap \{\mathrm{Im}(z) < 0\}$ and let be the arc $\nu := U \cap \{\mathrm{Im}(z) = 0\}$.

Theorem 9. *If function $f(z)$ is holomorphic in U_+, continuous in $U_+ \cup \nu$ and real on ν, then $\overline{f(\overline{z})}$ is holomorphic in U_- and*

$$F(z) := \begin{cases} f(z), & z \in U_+ \cup \nu \\ \overline{f(\overline{z})}, & z \in U_- \cup \nu \end{cases}$$

is holomorphic in U.

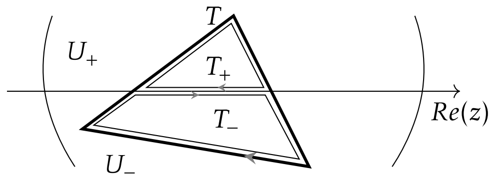
[2] By definition, characteristic function of a set S is $\chi(z \in S)$ which equals 1 if $z \in S$ and is 0 otherwise.

Proof. The assertion that $\overline{f(\bar{z})}$ is holomorphic in U_- is an easy consequence of Cauchy-Riemann equation.

Now, the assumptions on $f(z)$ imply that $F(z)$ is holomorphic in $U_+ \cup U_-$ and continuous in U. To prove that $f(z)$ is holomorphic in the entire U, it is enough to prove that

$$\int_T F(z)dz = 0$$

for any triangle T contained in U with its interior (this is the statement of Morera's theorem).



If T is contained entirely in U_+ and U_-, the integral is zero since $F(z)$ is holomorphic in $U_+ \cup U_-$. Now let T be a triangle intersecting ν. Then we can split it into to contours T_+ and T_- (see the picture), and the integral over both of them is zero. ☐