# Lecture 16

Let us start today's lecture by recalling an important result allowing to construct holomorphic functions as limits of holomorphic functions.

**Theorem 1** (Weierstrass' theorem)**.** *Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$, where $f_n$ is a function holomorphic in an open set $U_n$. Assume that $U_1 \subset U_2 \subset U_3 \subset \dots$ and let $U := \cup U_n$.*

*If $(f_n)_{n\in\mathbb{N}}$ converges to a limit function $f : U \to \mathbb{C}$ uniformly on every compact subset[1] of $U$, then*

- *$f(z)$ is holomorphic in $U$*
- *$f'(z) = \lim_{n\to\infty} f_n'(z)$ and convergence is uniform on every compact subset of $U$.*

*Proof.* First observe that for every compact $K \subset U$ union $\cup U_n$ is an open cover of $K$, hence there exists $N > 0$ such that $K \subset U_N \subset U$.

Take any $z_0 \in U$ and fix closed disk $\overline{B_R}(z_0)$. Take $N \in \mathbb{N}$ such that $\overline{B_R}(z_0) \subset U_n$ for $n \geqslant N$. In $\overline{B_R}(z_0)$, sequence $\{f_n\}_{n\geqslant N}$ uniformly converges to a function $f(z)$, hence for every loop $\gamma \subset B_R(z_0)$ we have

$$\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| \leqslant \text{Length}(\gamma)\cdot \sup_{z\in\gamma}|f_n(z) - f(z)|$$

Taking $z \to \infty$ and using the fact that $f_n$ being holomorphic implies $\int_\gamma f_n(z)dz = 0$ (Cauchy's theorem), we conclude that

$$\int_\gamma f(z)dz = 0$$

and $f(z)$ is holomorphic in $B_R(z_0)$ by Morera's theorem. Since $B_R(z_0) \subset U$ is arbitrary, $f(z)$ is holomorphic in the entire $U$. This proves the first assertion.

To get the uniform convergence of derivatives, we use Cauchy's formula for derivatives in $B_R(z_0)$:

$$f_n'(z) = \frac{1}{2\pi \boldsymbol{i}}\int_{|z-z_0|=R} \frac{f(\zeta)}{(\zeta - z)^2}d\zeta$$

therefore for $z \in B_{R/2}(z_0)$

$$|f'(z) - f_n'(z)| = \left|\frac{1}{2\pi \boldsymbol{i}}\int_{|z-z_0|=R} \frac{f(\zeta) - f_n(\zeta)}{(\zeta - z)^2}d\zeta\right| \leqslant \frac{1}{2\pi}\cdot\frac{1}{4R^2}\cdot \text{Length}(\gamma)\cdot \sup_{z\in B_R(z_0)}|f(z) - f_n(z)|.$$

As $n \to \infty$ the right hand side converges to 0 in $B_{R/2}(z_0)$. It remains to notice that any compact $K \subset U$ can be covered by a finite collection of disks $B_{R/2}(z_0)$ such that $B_R(z_0) \subset U$. □

**Example 2.** Let $\sum_0^\infty a_k z^k$ be an infinite power series with radius of convergence $R$. Then sequence of polynomials

$$f_n(z) = \sum_0^n a_k z^k$$

uniformly converges in the disk $B_R(0)$ and the limiting functions is holomorphic in this disk.

# Infinite series and products

## Partial fraction representation

Assume that function $f(z)$ is meromorphic in a connected open set $U$ and has poles $\{\zeta_k\}$. To every pole $\zeta_k$ of order $n_k$ we associate its *principle part*

$$P_k(z) := \sum_{i=1}^{n_k} \frac{a_i}{(z-\zeta_k)^i}.$$

[1] That is for every compact $K \subset U$ and every $\epsilon > 0$ there exists $N = N(K,\epsilon) \in \mathbb{N}$ such that $|f(z) - f_n(z)| < \epsilon$ for each $n > N$ and any $z \in K$.

If $f(z)$ has finitely many poles in $U$ we may write $f(z)$ as

$$f(z) = g(z) + \sum_k P_k(z), \tag{1}$$

where $g(z)$ is holomorphic in $U$.

If $f(z)$ has infinitely many poles in $U$, the above representation (1) includes infinite sum which may not be convergent. The point of Mittag-Leffler theorem is that, under extra assumptions, it is possible to modify the expression on the right-hand-side of (1) turning it into a convergent infinite sum.

**Theorem 3** (Mittag-Leffler theorem). *Let $\{\zeta_k\} \subset \mathbb{C}$ be a sequence such that $\lim_{k\to\infty} \zeta_k = \infty$ and let $\{P_k(z)\}$*

$$P_k(z) = \sum_{i=1}^{n_k} \frac{a_i}{(z-\zeta_k)^i}$$

*be an arbitrary collection of 'principle parts' of poles at $\zeta_k$. Then*

- *There exists a meromorphic function $f(z)$ in $\mathbb{C}$ with poles just at $\{\zeta_k\}$ and prescribed principle parts $P_k(z)$.*
- *Any such $f(z)$ can be written as*

$$f(z) = g(z) + \sum_k (P_k(z) - q_k(z))$$

  *where $g(z)$ is holomorphic in entire $\mathbb{C}$ and $\{q_k(z)\}$ are polynomials.*

*Proof.* Without loss of generality we assume that $0 \notin \{\zeta_k\}$. The idea is that instead of summing $P_k(z)$ which might produce a divergent infinite series, we can sum $P_k(z) - q_k(z)$, where $q_k(z)$ is the Taylor polynomial approximating $P_k(z)$ so that $P_k(z) - q_k(z)$ is the remainder term in the Taylor's series. By a smart choice of degrees of $\{q_k(z)\}$ we will guarantee that $\{P_k(z) - q_k(z)\}$ will form an absolutely convergent series in $\mathbb{C}$.

Not let us fill in the details. Let $q_k(z)$ be the polynomial part of the Taylor's formula for $P_k(z)$ at $z_0 = 0$ of degree $N_k$, where $N_k$ is a number to be chosen later:

$$P_k(z) = q_k(z) + \psi_{k,N_k}(z) z^{N_k+1}.$$

We now that the remained $\psi_{k,N_k}$ has an integral representation

$$\psi_{k,N_k}(z) = \frac{1}{2\pi \boldsymbol{i}} \int_C \frac{P_k(\zeta)}{\zeta^{N_k+1}(\zeta - z)} d\zeta.$$

If we choose $C = \{|z| = \frac{|\zeta_k|}{2}\}$, the above representation will hold in the disk $B_{|\zeta_k|/2}(0)$. Moreover, for $|z| < |\zeta_k|/4$ we have the following bound:

$$|\psi_{k,N_k}(z)| \leqslant \frac{1}{2\pi} \underbrace{\frac{2\pi|\zeta_k|}{2}}_{\text{Length}(\gamma)} \frac{M_k}{(|\zeta_k|/2)^{N_k+1} \frac{|\zeta_k|}{4}},$$

where $M_k = \sup_{|z|=|\zeta_k|/2} P_k(z)$.

Hence

$$|P_k(z) - q_k(z)| \leqslant 2M_k \left(\frac{2|z|}{|\zeta_k|}\right)^{N_k+1}.$$

Let us choose $N_k$ large enough so that $M_k \leqslant 2^{N_k - k}$. Then, as long as $|z| < |\zeta_k|/4$, we have

$$|P_k(z) - q_k(z)| \leqslant 2^{-k}.$$

Now, consider any disk $B_R(0)$. Then

$$\sum_{|\zeta_k|/4 > R} (P_k(z) - q_k(z))$$

uniformly converges in $\overline{B}_R(0)$ to a holomorphic function in $\overline{B}_R(0)$, and

$$\sum_k (P_k(z) - q_k(z)) = \sum_{|\zeta_k|/4 \leqslant R} (P_k(z) - q_k(z)) + \sum_{|\zeta_k|/4 > R} (P_k(z) - q_k(z))$$

is well-defined meromorphic function in $B_R(0)$ with principle parts $P_k(z)$ at $\zeta_k$. Since $R$ is arbitrary, $h(z) := \sum_k (P_k(z) - q_k(z))$ is a well-defined meromorphic function in $\mathbb{C}$ with prescribed principle parts of poles at $\zeta_k$. This proves the first statement.

To prove the second statement, consider any meromorphic function $f(z)$ with prescribed poles and principle parts. Then $g(z) := f(z) - h(z)$ is holomorphic in the entire $\mathbb{C}$. □

**Example 4.** Consider function $f(z) = \pi \cot \pi z = \pi \frac{\cos \pi z}{\sin \pi z}$. This function has poles at all points $\zeta_n = n, n \in \mathbb{Z}$ with principle part $P_n(z) = \dfrac{1}{z+n}$. Naively, we would write

$$\pi \cot \pi z \text{ `` = '' } g(z) + \sum_{n\in\mathbb{Z}} \frac{1}{z+n}.$$

Unfortunately, the above infinite sum needs to be properly understood. One way to interpret it is to write it as

$$h(z) := \frac{1}{z} + \sum_{n\in\mathbb{Z}, n\neq 0} \left( \frac{1}{z+n} - \frac{1}{n} \right).$$

Then the new infinite sum above converges uniformly in every closed disk $\overline{B_R(0)}$, since

$$\left| \frac{1}{z+n} - \frac{1}{n} \right| \leqslant \frac{2R}{n^2}$$

for $n$ large enough.

Therefore both $\pi \cot \pi z$ and $h(z)$ represent meromorphic functions in $\mathbb{C}$ with the same set of poles and principle parts at these poles. Hence

$$g(z) := \pi \cot \pi z - h(z)$$

is an entire holomorphic function.

We claim that $g(z) = 0$. To prove it, we will show that $g(z)$ is bounded and must be a constant by Liouville theorem. Since both $g(z)$ and $\pi \cot \pi z$ are odd, this constant must be 0.

First, note that both $\pi \cot \pi z$ and $h(z)$ are periodic as $z \mapsto z+1$. Hence it is enough to prove boundedness in the strip $|\mathfrak{Re}(z)| \leqslant 1/2$. For $\mathfrak{Im}(z) > 1$, we have

$$\cot \pi z = \boldsymbol{i} \frac{e^{\boldsymbol{i}\pi z} + e^{-\boldsymbol{i}\pi z}}{e^{\boldsymbol{i}\pi z} - e^{-\boldsymbol{i}\pi z}} = \boldsymbol{i} \frac{e^{-2\pi y} + e^{-2\pi \boldsymbol{i} x}}{e^{-2\pi y} - e^{-2\pi \boldsymbol{i} x}}$$

which is clearly bounded.

It remains to prove that $h(z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}, n\neq 0} \left( \frac{1}{z+n} - \frac{1}{n} \right)$. is bounded in the same region. Indeed, we can rewrite

$$\sum_{n\in\mathbb{Z}, n\neq 0} \left( \frac{1}{z+n} - \frac{1}{n} \right) = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

so that

$$\left| \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \right| \leqslant C \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} < C \int_0^{\infty} \frac{y}{y^2 + x^2} dx = C \frac{\pi}{2}.$$

So, $g(z)$ is bounded for $\mathfrak{Im}(z) > 1$. Similarly, it is bounded for $\mathfrak{Im}(z) < -1$. It remains to note that $g(z)$ being continuous, is bounded in $|\mathfrak{Im}(z)| \leqslant 1$, $|\mathfrak{Re}(z)| \leqslant 1/2$. Hence $g(z)$ is bounded in the entire $\mathbb{C}$ and must be a constant.

$$\pi \cot \pi z = \lim_{N\to\infty} \sum_{|n|\leqslant N} \frac{1}{z+n}.$$

## Infinite products

Before discussing infinite products of holomorphic functions, we will need some preliminaries on infinite products of numbers.

**Definition 5.** Given a sequence of complex numbers $\{a_n\}$, we say that the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

*converges* if the limit

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)$$

of the partial products exists.

**Proposition 6.** *If $\sum|a_n|<\infty$ then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges. Moreover, the product converges to* 0 *if and only if one of the factors is* 0*.*

*Proof.* If $\sum|a_n|$ converges, then for $n$ large enough $|a_n|<1/2$. Disregarding, if necessary, the initial terms, we may assume that for all $n\in\mathbb{N}$ we can define $\mathrm{Log}(1+a_n)$ by its principle branch. Hence, the partial products are

$$\prod_{n=1}^{N}(1+a_n)=\prod_{n=1}^{N}e^{\mathrm{Log}(1+a_n)}=:e^{B_N},$$

where $B_N=\sum_{n=1^N}\mathrm{Log}(1+a_n)$. Since for $|z|\leqslant 1/2$ we have $|\mathrm{Log}(1+z)|\leqslant 2|z|$,we can conclude that the series defining $B_N$ absolutely converges. If $B:=\lim B_N$, by the continuity of Log we find

$$\prod_{n=1}^{N}(1+a_n)\to e^{B}.$$

The resulting limit is nonzero, unless we have omitted a zero factor in the very beginning. □

**Definition 7.** The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to be *absolutely convergent* iff $\sum\mathrm{Log}(1+a_n)$ is absolutely convergent.

**Proposition 8.** *The product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent iff $\sum|a_n|$ converges.*

*Proof.* In either case we necessarily have $a_n\to 0$. Hence for $n$ large enough we have

$$\frac{1}{2}|a_n|<|\mathrm{Log}(1+a_n)|<\frac{3}{2}|a_n|.$$

Therefore

$$\sum|a_n|\text{ converges }\iff\sum\mathrm{Log}(1+a_n)\text{ converges }.$$

□