Lecture 17

Infinite products

Before discussing infinite products of holomorphic functions, we will need some preliminaries on infinite products of numbers.

Definition 1. Given a sequence of complex numbers $\{a_n\}$, we say that the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges if the limit

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n)$$

of the partial products exists.

Often it is additionally assumed that all but finitely many factors are non-zero, and then the convergence is considered for the product of these non-zero factors.

Proposition 2. *If $\sum |a_n| < \infty$ then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges. Moreover, the product converges to 0 if and only if one of the factors is 0.*

Proof. If $\sum |a_n|$ converges, then for n large enough $|a_n| < 1/2$. Disregarding, if necessary, the initial terms, we may assume that for all $n \in \mathbb{N}$ we can define $\mathrm{Log}(1+a_n)$ by its principle branch. Hence, the partial products are

$$\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\mathrm{Log}(1+a_n)} =: e^{B_N},$$

where $B_N = \sum_{n=1}^{N}\mathrm{Log}(1+a_n)$. Since for $|z| \leqslant 1/2$ we have $|\mathrm{Log}(1+z)| \leqslant 2|z|$,we can conclude that the series defining B_N absolutely converges. If $B := \lim B_N$, by the continuity of Log we find

$$\prod_{n=1}^{N}(1+a_n) \to e^B.$$

The resulting limit is nonzero, unless we have omitted a zero factor in the very beginning. □

Definition 3. The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to be *absolutely convergent* iff $\sum \mathrm{Log}(1+a_n)$ is absolutely convergent.

Proposition 4. *The product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent iff $\sum |a_n|$ converges.*

Proof. In either case we necessarily have $a_n \to 0$. Hence for n large enough we have

$$\frac{1}{2}|a_n| < |\mathrm{Log}(1+a_n)| < \frac{3}{2}|a_n|.$$

Therefore

$$\sum |a_n| \text{ converges } \iff \sum \mathrm{Log}(1+a_n) \text{ converges absolutely }.$$

□

Weierstrass factorization theorem

If $f(z)$ is a nowhere zero entire function, then there is a well-defined logarithm of $f(z)$, i.e., an entire function $g(z)$ such that

$$e^{g(z)} = f(z).$$

More generally, if $f(z)$ has finitely many zeros[1] $\{a_k\}_{k=1}^N$ away from the origin, then we can factor $f(z)$ as

$$f(z) = z^M e^{g(z)} \prod_{k=1}^{N} \left(1 - \frac{z}{a_k}\right) \tag{1}$$

where $g(z)$ is an entire function. Today we will address the following question:

Question. *To what extent can we generalize (1) if $f(z)$ has infinitely many zeros in $\mathbb{C}$?*

From now on we assume that $f(z)$ has infinitely many zeros $\{a_k\}$ apart from 0. If product $\prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k}\right)$ converges uniformly on compact subsets of $\mathbb{C}$, then it defines an entire function with zeros only at $\{a_k\}$. Therefore, we can write

$$\frac{f(z)}{\prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k}\right)} = z^M e^{g(z)}$$

where M is the order of zero of $f(z)$ at $z = 0$ and $g(z)$ is some entire function. Equivalently

$$f(z) = z^M e^{g(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right).$$

To ensure that the product $\prod_{k=1}^{\infty}\left(1 - \frac{z}{a_k}\right)$ uniformly converges on every compact set it is enough to assume that the series $\sum \frac{1}{|a_k|}$ converges. Indeed, on every disk $B_R(0)$ for k large enough we have

$$|\log(1 - z/a_k)| < R/|a_k|,$$

so the sum of logarithms absolutely converges.

To find factorization for $f(z)$ in general, we have to modify factors of infinite product to make it absolutely convergent. This can be done by a trick similar to the one used in the proof of Mittag-Leffler theorem.

Theorem 5. *There exists an entire function with arbitrarily prescribed zeros $\{a_k\}$, as long as $a_k \to \infty$ if the numbers of zeros is infinite. Moreover, every entire function with zeros exactly at $\{a_k\}$ can be written as*

$$f(z) = z^M e^{g(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{\frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \dots + \frac{1}{N_k}\left(\frac{z}{a_k}\right)^{N_k}}.$$

where the product is taken over all $a_k \neq 0$, $\{N_k\}$ are integers and $g(z)$ is an entire function.

Proof. We plan to show that there exists a sequence of polynomials $q_k(z)$ such that the product

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) e^{q_k(z)}$$

uniformly converges on compact sets.

We will find polynomials such that the infinite sum

$$\sum_{k=1}^{\infty} (\mathrm{Log}(1 - z/a_k) + p_k(z)) := \sum_{k=1}^{\infty} \psi_k(z).$$

uniformly converges on the compact sets. Given $z \in B_R(0)$, for a fixed R the principle branch of logarithm is well-defined for k large enough, since $a_k \to \infty$.

As for the proof of Mittag-Leffler theorem, we will choose $-p_k(z)$ to be the initial segment of the Taylor's series of $\mathrm{Log}(1 - z/a_k)$. Namely, as long as $|z/a_k| < 1$, we have

$$\mathrm{Log}\left(1 - \frac{z}{a_k}\right) = -\frac{z}{a_k} - \frac{1}{2}\left(\frac{z}{a_k}\right)^2 - \dots - \frac{1}{N_k}\left(\frac{z}{a_k}\right)^{N_k} + \psi_k(z),$$

where $N_k \in \mathbb{N}$ to be chosen later.

[1] If a_i is a zero of order n_i, then we assume that a_i appears exactly n_i times in the sequence $\{a_k\}$.

As before, for $z \in B_R(0)$ we have an estimate for the remainder term in the Taylor's formula:

$$\psi_k(z) = -\frac{1}{N_k+1}\left(\frac{z}{a_k}\right)^{N_k+1} - \frac{1}{N_k+2}\left(\frac{z}{a_k}\right)^{N_k+2} - \dots$$

so

$$|\psi_k(z)| \leqslant \frac{1}{N_k+1}\left(\frac{R}{|a_k|}\right)^{N_k+1} \frac{1}{1-\frac{R}{|a_k|}}.$$

Now we choose $\{N_k\}$ such that the infinite series

$$\sum \frac{1}{N_k+1}\left(\frac{R}{|a_k|}\right)^{N_k+1}$$

converges, e.g., by taking $N_k = k$. Then the infinite product

$$h(z) := \prod_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)e^{q_k(z)}$$

with

$$q_k(z) = \frac{z}{a_k} + \frac{1}{2}\left(\frac{z}{a_k}\right)^2 + \dots + \frac{1}{N_k}\left(\frac{z}{a_k}\right)^{N_k}$$

converges on compact sets and defines an entire function with zeros at a_k.

Hence if $f(z)$ is any other function with the same zeros (and multiplicities) besides $z = 0$, then

$$f(z) = z^M e^{g(z)} h(z).$$

□

Corollary 6. *Any meromorphic function in $\mathbb{C}$ is a quotient of two holomorphic functions.*

Proof. Assume that function $f(z)$ is meromorphic with poles at $\{\zeta_k\}$. Construct a holomorphic function $h(z)$ with zeros at $\{\zeta_k\}$ and the same orders as the orders of poles of $f(z)$. Then

$$g(z) := f(z)h(z)$$

is holomorphic in entire plane $\mathbb{C}$, so

$$f(z) = \frac{g(z)}{h(z)}.$$

□

Example 7. Consider a function $f(z) = \sin \pi z$. It has zeros of order 1 at all integer numbers. It is not hard to check (see homework) that the product

$$\prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}$$

is uniformly convergent on compact sets. Therefore,

$$\sin \pi z = z e^{g(z)} \prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n}$$

for some entire function $g(z)$.

Taking logarithmic derivatives of both sides, we find

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right).$$

From the previous class we know that $g'(z) = 0$, so $g(z) = const$. It remains to use the fact that $\sin(\pi z)/z \to 0$ as $z \to 0$ to conclude that

$$\sin \pi z = \pi z \prod_{n\neq 0}\left(1-\frac{z}{n}\right)e^{z/n} = \pi z \prod_{z\in\mathbb{N}}\left(1-\frac{z^2}{n^2}\right),$$

where in the last identity we paired together factors corresponding to n and $-n$.

Genus and order of entire function

It is often possible to choose powers N_k in the product

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)e^{q_k(z)}, \qquad q_k(z)=\frac{z}{a_k}+\frac{1}{2}\left(\frac{z}{a_k}\right)^2+\cdots+\frac{1}{N_k}\left(\frac{z}{a_k}\right)^{N_k}$$

to be constant $N_k = h$ for all k. From the proof of Weiestrass theorem, we saw that in this case the series

$$\sum \frac{1}{h+1}\left(\frac{R}{|a_k|}\right)^{h+1}$$

must be convergent for any R, that is $\sum 1/|a_k|^{h+1} < \infty$. The smallest h for each it holds is called *the genus* of the product

$$\prod_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)e^{q_k(z)}$$

and the product is called *the canonical product*.

Definition 8. If function $f(z)$ can be represented as

$$f(z)=z^M e^{g(z)}\prod_{k=1}^{\infty}\left(1-\frac{z}{a_k}\right)e^{q_k(z)}.$$

where $g(z)$ and each $q_k(z)$ are polynomials of degree $\leqslant h$, and h is the smallest number for each such expression is possible, then h is called *the genus* of $f(z)$.

Example 9. From the infinite product for $\sin \pi z$ we see that is has genus 1. Any polynomial has genus zero.

Informally, the genus of canonical product controls how quickly its zeros escape to infinity.

Denote by $M(r)$ the maximum of $|f(z)|$ on $\{|z| = R\}$.

Definition 10. The *order* of an entire function $f(z)$ is defined as

$$\lambda := \limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}.$$

Equivalently it is the smallest number such that

$$M(R) \leqslant e^{R^{\lambda+\epsilon}}$$

for all $\epsilon > 0$ and all R sufficiently large.

The genus and order of an entire function are closely related to each other. We leave the following theorem without a proof.

Theorem 11 (Hadamard Factorization Theorem)**.** *The genus and order of an entire function satisfy*

$$h \leqslant \lambda \leqslant h+1.$$

Roughly speaking, Hadamard theorem provide control of the growth rate of $|f(z)|$ in terms of the growth of the number of zeros of $f(z)$ in $B_R(0)$.