Lecture 18

Special functions

Gamma function

Euler's Gamma function $\Gamma(z)$ is one of the key *special* meromorphic functions in complex analysis. Its significance is supported by many applications combinatorics, number theory, differential equations, probability, and many other areas.

Theorem 1. *There exists a unique function $\Gamma(s)$ of a complex variable $s \in \mathbb{C}$ such that*

1. *$\Gamma(s)$ is meromorphic in $\mathbb{C}$;*
2. *$\Gamma(n+1) = n!$ for $n \in \mathbb{N}$;*
3. *$\Gamma(1/2) = \sqrt{\pi}$;*
4. *$\Gamma(s)$ has integral representation:*
$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx \quad (\mathfrak{Re}(s) > 0);$$
5. *$\Gamma(s)$ has hybrid integral representation*
$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-x} x^{s-1} dx \quad (s \in \mathbb{C});$$
6. *$\Gamma(s)^{-1}$ has infinite product representation:*
$$\Gamma(s)^{-1} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \quad (s \in \mathbb{C});$$
7. *$\Gamma(s)$ is the limit of finite products*
$$\Gamma(s) = \lim_{n \to \infty} \frac{n! n^s}{s(s+1)\dots(s+n)} \quad (s \in \mathbb{C});;$$
8. *$\Gamma(s)$ has no **zeros**;*
9. *$\Gamma(s)$ has simple **poles** at $s = 0, -1, -2, \dots$ with $\mathrm{res}_{-n}(\Gamma) = \frac{(-1)^n}{n!}$;*
10. *$\Gamma(s)$ satisfies **functional equation***
$$\Gamma(s+1) = s\Gamma(s) \quad (s \in \mathbb{C});;$$
11. *$\Gamma(s)$ satisfies reflection formula*
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad (s \in \mathbb{C});$$

We will prove most of the above properties. The remaining ones are left as an exercise.

Proof. **Definition for $\mathfrak{Re}(s) > 0$ (integral representation).**

We start with the definition of $\Gamma(s)$ for $\mathfrak{Re}(s) > 0$ and deduce from it *analytic continuation* of $\Gamma(s)$ to a meromorphic function in the entire $\mathbb{C}$ and all of its properties.

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx. \tag{1}$$

First, note that this integral converges for $s \in \mathbb{R}, s > 0$ since near $x = 0$ function x^{s-1} is integrable, and for large x the integrand has exponential decay.

To prove that (1) defines an analytic function for all $\mathfrak{Re}(s) > 1$ we argue as follows. For every n there is a holomorphic function

$$F_n(s) := \int_{1/n}^{n} e^{-x} x^{s-1} dx$$

and in every strip $\{\delta < \mathfrak{Re}(s) < M\}$ with $s := \sigma + \boldsymbol{i}t$ we have

$$|\Gamma(s) - F_n(s)| \leqslant \int_0^{1/n} e^{-x} x^{\sigma-1} dx + \int_n^{\infty} e^{-x} x^{\sigma-1} dx < \frac{(1/n)^{\delta}}{\delta} + C(M)e^{-n/2}$$

which uniformly goes to 0 as $n \to \infty$.

Functional equation for $\mathfrak{Re}(s) > 0$.

Lemma 2. *For $\mathfrak{Re}(s) > 0$*

$$\Gamma(s+1) = s\Gamma(s).$$

In particular, $\Gamma(n+1) = n!$, $n \in \mathbb{Z}$.

Proof of Lemma. We can integrate by parts:

$$\int_{1/n}^{n} \frac{d}{dx}(e^{-x}x^s)dx = -\int_{1/n}^{n} e^{-x} x^s dx + s\int_{1/n}^{n} e^{-x} x^{s-1} dx.$$

It remains to let $n \to \infty$ and note that the left-hand side goes to zero, since $e^{-x}x^s$ goes to zero as $x \to 0$ or $+\infty$. Finally $\Gamma(1) = \int_0^{\infty} e^{-x} dx = 1 = 0!$ and the rest follows by induction. □

Analytic continuation.

We can use functional equation $\Gamma(s+1) = s\Gamma(s)$ to continue $\Gamma(s)$ to a meromorphic function of $s \in \mathbb{C}$. Specifically, we first define a function

$$F_1(s) = \frac{\Gamma(s+1)}{s}.$$

This is a meromorphic function of s, $\mathfrak{Re}(s) > -1$ with a single pole at $s = 0$. Moreover, it coincides with $\Gamma(s)$ for all $s > 0$ due to the functional equation.

We can iterate this process and define

$$F_m(s) := \frac{\Gamma(s+m)}{s(s+1)\dots(s+m-1)}.$$

This function is meromorphic for $\mathfrak{Re}(s) > -m$, coincides with $\Gamma(s)$ (and all previously defined $F_i(s)$) on its domain and has simple poles at $s = 0, -1, \dots, -(m-1)$.

Eventually, we extend $\Gamma(s)$ to a meromorphic function in the entire plane.

Remark 3. The extension of $\Gamma(s)$ to $\mathbb{C}$ still satisfies the functional equation $\Gamma(s+1) = s\Gamma(s)$.

Hybrid integral representation.

The reason why the integral representation (1) works only for $\mathfrak{Re}(s) > 0$ and all s is that the integral improper integral diverges near $x = 0$. To *isolate* this issue we can rewrite the definition of $\Gamma(s)$ as follows:

$$\Gamma(s) = \int_0^1 e^{-x} x^{s-1} dx + \int_1^{\infty} e^{-x} x^{s-1} dx$$

The latter integral defines an entire analytic function. Expanding e^{-x} we also rewrite the former integral as

$$\int_0^1 e^{-x} x^{s-1} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)}$$

Therefore

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)} + \int_1^{\infty} e^{-x} x^{s-1} dx.$$

The advantage of this presentation is that it defines a meromorphic function for all $s \in \mathbb{C}$.

Finite product limit

Lemma 4. *For $\mathfrak{Re}(s) > 0$ we have*

$$\Gamma(s) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1} dx.$$

Proof of lemma. The integrand monotonically increases to e^{-x} as $n \to \infty$ therefore we have the result by dominated converges. □

It is an elementary exercise to prove that

$$\int_0^n \left(1 - \frac{x}{n}\right)^n x^{s-1} dx = n^s \int_0^1 (1-t)^n t^{s-1} dt = \frac{n! n^s}{s(s+1)\dots(s+n)}.$$

Infinite product representation for $\Gamma(s)^{-1}$ with $\mathfrak{Re}(s) > 0$

$$\Gamma^{-1}(s) = \lim_{n\to\infty} \frac{s(s+1)\dots(s+n)}{n! n^s} = s \lim_{n\to\infty} e^{-s\log n} \left(1 + \frac{s}{1}\right) \cdots \left(1 + \frac{s}{n}\right) = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n},$$

where $\gamma := \lim_{n\to\infty}(1 + 1/2 + \cdots + 1/n - \log n)$ is the Euler-Mascheroni constant.

The above representation is well-defined in the entire $\mathbb{C}$. In fact, this is the Weierstrass factorization of the entire function $\Gamma(s)^{-1}$. In particular, we see that $\Gamma(s)^{-1}$ has genus 1. □

Volume of an n-dimensional ball

The values of the Gamma function at integers and half-integers naturally occur in the formula for the volumes of n-dimensional balls and spheres.

Theorem 5. *Let $A_{n-1}(r)$ be a surface area of the $(n-1)$-dimensional sphere: $S^{n-1}(r) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| = r\}$ and $V_n(r)$ the volume of the n-dimensional ball of radius r. Then*

$$A_{n-1}(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}.$$

$$V_n(r) = \frac{r}{n} A_{n-1}(r) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n.$$

Proof. Consider function $f(\mathbf{x}) = \exp(-|\mathbf{x}|^2/2)$, where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} = \prod_{k=1}^{n} \left(\int_{-\infty}^{+\infty} e^{-x_k^2/2} dx_k \right) = (\sqrt{2\pi})^n.$$

On the other hand, since $f(\mathbf{x})$ is rotationally symmetric, we find:

$$\int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} = \int_0^{+\infty} e^{-r^2/2} A_{n-1}(r) dr$$

Now, due to scaling properties, $A_{n-1}(r) = A(1) r^{n-1}$. Hence using substitution $t = r^2/2$ we can compute

$$\int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x} = A_{n-1}(1) \int_0^{+\infty} e^{-r^2/2} r^{n-1} dr = 2^{n/2-1} A_{n-1}(1) \int_0^{\infty} e^{-t} t^{n/2-1} dt = 2^{n/2-1} A_{n-1}(1) \Gamma(n/2).$$

Comparing values of $\int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x}$ computed in two different ways, we find

$$A_{n-1}(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Finally, for the volume we have:

$$V_n(r) = \int_0^r A_{n-1}(r) dr = A(1) \int_0^r r^{n-1} dr = \frac{\pi^{n/2}}{\Gamma(n/2+1)} r^n.$$

□

Remark 6. Using the special value $\Gamma(1/2) = \sqrt{\pi}$, the above formulas could be rewritten in a slightly different manner for odd n.

We also observe something remarkable. The volumes of the unit balls in $\mathbb{R}^n$ increase for $n \leqslant 5$, but then decrease to 0 as $n \to \infty$.