Lecture 19

Zeta function $\zeta(s)$

Definition 1. For $s = \sigma + \boldsymbol{i}t$, $\sigma > 1$ define

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $n^s := e^{s \log n}$ is defined using the principle branch of the logarithm. For any $\sigma > 1 + \delta$, $\delta > 0$ we have

$$\left| \frac{1}{n^s} \right| < \frac{1}{n^{1+\delta}}.$$

Since $\sum 1/n^{1+\delta} < \infty$, the power series defining $\zeta(s)$ is absolutely and uniformly convergent in any region of the form $\{\sigma > 1 + \delta\}$.

Function $\zeta(s)$ is called *Riemann zeta function*.

Proposition 2 (Euler product formula)**.** *For $\mathfrak{Re}(s) > 1$ Zeta function has infinite product representation*

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} \tag{1}$$

where the product is taken over all positive prime numbers p.

Proof. Consider s in the region $\mathfrak{Re}(s) > 1 + \delta$, $\delta > 0$.

First, we note that the product $\prod\limits_p \left(1 - \frac{1}{p^s}\right)$ is uniformly and absolutely convergent, since

$$\sum |p^{-s}| < \sum n^{-1-\delta} < \infty.$$

Next, if we let $p_1, \ldots, p_N$ to be first N primes, then

$$\zeta(s)(1 - p_1^{-s}) \ldots (1 - p_N^{-s}) = \sum_{n'} \frac{1}{(n')^s}$$

where the sum is taken over all n' such that prime factorization of n' is free of $p_1, \ldots, p_N$. In particular, if $N(M)$ is large enough, this sum will not include first M natural numbers. In particular

$$|\zeta(s)(1 - p_1^{-s}) \ldots (1 - p_N^{-s})| < \sum_{n=M+1}^{\infty} \frac{1}{n^{1+\delta}}.$$

Taking M large, the latter tail can be made arbitrary small. □

Remark 3. Formally (1) is just a reformulation of the Fundamental Theorem of Arithmetics, since

$$\frac{1}{1 - p^{-s}} = \sum_k \frac{1}{p^{ks}}$$

and the product of series $\sum_k \frac{1}{p^{ks}}$ over all primes p after expanding just picks ups every term n^{-s} exactly once.

Corollary 4. *Zeta function has no zeros in $\mathfrak{Re}(s) > 1$*[1].

[1] Later we will show that $\zeta(s)$ can be extended to meromorphic function in $\mathbb{C}$ and has no zeros on $\mathfrak{Re}(s) = 1$. This is a *toy* version of Riemann hypothesis.

Analytic continuation of $\zeta(s)$

The first difficulty which we have to overcome is to define analytic continuation of $\zeta(s)$ to a larger domain. First, let us explain how one can extend $\zeta(s)$ to a *meromorphic* function in $\mathfrak{Re}(s) > 0$.

To do so, we manipulate with the definition of $\zeta(s)$ in $\mathfrak{Re}(s) > 1$ and rewrite it in a way which would make sense in $\mathfrak{Re}(s) > 0$.

Lemma 5. *For $\mathfrak{Re}(s) > 1$ we have*

$$\zeta(s) = \frac{1}{s-1} - \int_1^\infty (x^{-s} - [x]^{-s})\,dx$$

where $[x]$ is the floor (integer part) of a real number x.

Proof. Since for $\mathfrak{Re}(s) > 1 + \delta$ all the series and integrals below are absolutely and uniformly convergent, we have

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^\infty \frac{1}{n^s} = \sum_{n=1}^\infty \left(\int_n^{n+1} \frac{dx}{x^s} + \left(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}\right)\right)\\ &= \int_1^\infty \frac{dx}{x^s} + \sum_{n=1}^\infty \left(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}\right)\\ &= \frac{1}{s-1} - \int_1^\infty (x^{-s} - [x]^{-s})\,dx.\end{aligned}$$

□

The key feature of the above expression is that it makes sense as long as $\mathfrak{Re}(s) > 0$. Indeed, by mean value theorem we can bound

$$|x^{-s} - [x]^{-s}| < |s| x^{-\mathfrak{Re}(s)-1}$$

which shows that the integral is absolutely and uniformly convergent in any set $\{\mathfrak{Re}(s) > \delta\} \cap \{|s| < R\}$.

Corollary 6. *Function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.*

To extend $\zeta(s)$ further, we study the product $\Gamma(s)\zeta(s)$.

Proposition 7. *For $\mathfrak{Re}(s) > 1$*

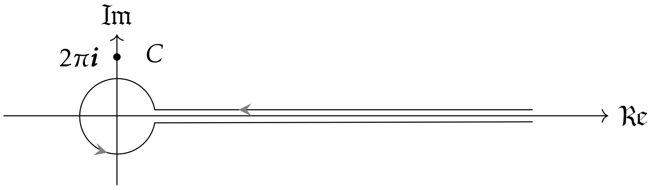
$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx.$$

Proof. Homework exercise. □

Theorem 8. *For $\mathfrak{Re}(s) > 1$*

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi \boldsymbol{i}} \int_C \frac{(-z)^{s-1}}{e^z - 1}\,dz$$

where C is the contour below.



Proof. First, we note the poles of the integrand are at $2k\pi \boldsymbol{i}$, and as as long as the circle does not include these poles, the integral does not depend on the choice of the circle. Moreover, as the size of the circle goes to 0, the corresponding integral also goes to zero.

Next, integrals over the two rays are computed using different branches of $(-z)^{s-1}$. Namely on the upper edge $(-z)^{s-1} = x^{s-1} e^{-(s-1)\pi \boldsymbol{i}}$ and on the lower edge $(-z)^{s-1} = x^{s-1} e^{(s-1)\pi \boldsymbol{i}}$.

Hence, we obtain

$$\int_C \frac{(-z)^{s-1}}{e^z-1}dz = -\int_0^\infty \frac{x^{s-1}e^{-(s-1)\pi \boldsymbol{i}}}{e^x-1}dx + \int_0^\infty \frac{x^{s-1}e^{(s-1)\pi \boldsymbol{i}}}{e^x-1}dx$$
$$=2\boldsymbol{i}\sin(\pi(s-1))\zeta(s)\Gamma(s).$$

Finally, as $\sin(\pi(s-1)) = -\sin(\pi)$ and $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we get the stated formula. □

The importance of this new representation is that the integral on the right hand side yields an *entire* function, i.e., it is holomorphic function of $s \in \mathbb{C}$. Therefore, $\zeta(s)$ being a product of meromorphic function $\Gamma(1-s)$ and an entire function is itself a meromorphic function of $s \in \mathbb{C}$.

Corollary 9 (Pole of $\zeta(s)$)**.** *Function $\zeta(s)$ has a unique pole at $s = 1$.*

Proof. Potentially poles of $\zeta(s)$ can only occur at poles of $\Gamma(1-s)$, i.e, $1, 2, 3, \ldots$. But we already know that $\zeta(s)$ is holomorphic in $\mathfrak{Re}(s) > 1$. □

Corollary 10 (Values at negative integers)**.** *For $m \in \mathbb{N}$ we have*

$$\zeta(-2m) = 0, \qquad \zeta(-2m+1) = (-1)^m \frac{B_m}{2m},$$

where B_m are Bernoulli numbers.

Zeros at negative even integers are called *trivial zeros* of $\zeta(s)$.

Proof. From the formula for $\zeta(s)$, we have

$$\zeta(-n) = (-1)^n \frac{n!}{2\pi \boldsymbol{i}} \int_C \frac{z^{-n-1}}{e^z-1}dz.$$

It remain to notice that the the integral computes $1/2\pi\boldsymbol{i}$ times the coefficient at z^n in Laurent expansion of $1/(e^z-1)$, which is by one of the homework problems

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_k}{(2k)!} z^{2k-1}.$$

□

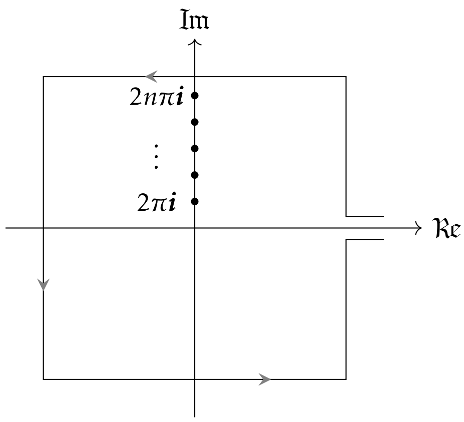
Reflection formula for $\zeta(s)$

It was discovered by Riemann that the values $\zeta(s)$ and $\zeta(1-s)$ are related by an explicit equation. This provides a good control over $\zeta(s)$ for $\mathfrak{Re}(s) < 0$ and the key to understanding zeta function lies in its behavior in the *critical strip* $0 \leqslant \mathfrak{Re}(s) \leqslant 1$.

Theorem 11 (Functional equation)**.**

$$\zeta(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$

Proof. To prove this identity we introduce a contour C_n.



Consider a closed contour $C_n - C$ (where C is the contour introduced above, so that infinite rays "cancel out"). It has winding number 1 around each of the poles $\pm 2k\pi\boldsymbol{i}$, $k = 1, \dots n$ of $(-z)^{s-1}/(e^z - 1)$. Therefore, by residue theorem we have

$$\frac{1}{2\pi\boldsymbol{i}} \int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1} dz = \sum_{k=1}^{n} [(-2k\pi\boldsymbol{i})^{s-1} + (2k\pi\boldsymbol{i})^{s-1}] = 2 \sum_{k=1}^{n} (2k\pi)^{s-1} \sin \frac{\pi s}{2}.$$

Using an elementary estimate, we find that for $\mathfrak{Re}(s) < 0$ the integral over C_n goes to 0 as $n \to \infty$, therefore

$$\lim_{n\to\infty} \int_{C_n - C} \frac{(-z)^{s-1}}{e^z - 1} dz = -\int_C \frac{(-z)^{s-1}}{e^z - 1} dz.$$

Hence as long as $\mathfrak{Re}(s) < 0$, we conclude

$$\frac{\zeta(s)}{\Gamma(1-s)} = 2 \cdot (2\pi)^{s-1} \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} n^{s-1}$$

Since this identity holds for $\mathfrak{Re}(s) < 0$ and both functions are meromorphic in $\mathbb{C}$, it must hold for all s. □

Corollary 12 (Values at positive even integers)**.**

$$\zeta(2m) = \frac{(-1)^{m-1}(2\pi)^{2m}}{2(2m)!} B_m$$

Proof. Using the reflection formula for the Gamma function, we can rewrite the functional equation for $\zeta(s)$ as

$$\zeta(1-s) = 2^{1-s}\pi^{-s} \cos \frac{\pi s}{2} \Gamma(s)\zeta(s).$$

Setting $s = 2m$ and using the known values of $\zeta(1-2m)$, we get the stated formula. □

Remark 13. While explicit values of ζ at positive even integers were computed by Euler, almost nothing is known about numbers $\zeta(2m+1)$. In particular, it is not even known if all of them are irrational.

An equivalent way to express the functional equation for $\zeta(s)$ is to consider function

$$\xi(s) := \frac{s(1-s)}{2} \pi^{-s/2} \Gamma(s/2)\zeta(s),$$

then using Legendre's duplication formula (HW#3)

$$\xi(s) = \xi(1-s).$$

Remark 14. Functional equation for $\zeta(s)$ implies that its *non-trivial* are located in the strip $0 \leqslant \mathfrak{Re}(s) \leqslant 1$ and are symmetric across $\mathfrak{Re}(s) = 1/2$ axis. The statement of the famous *Riemann conjecture* is that all non-trivial zeros are located on the axis $\mathfrak{Re}(s) = 1/2$.

We will prove the following *baby* version of Riemann Hypothesis.

Proposition 15. *Function $\zeta(s)$ does not have zeros on the line $\mathfrak{Re}(s) = 1$*

Lemma 16. *For any $s = \sigma + \boldsymbol{i}t$ with $\sigma > 0$. we have*

$$|\zeta(\sigma)^3 \zeta(\sigma + \boldsymbol{i}t)^4 \zeta(\sigma + 2\boldsymbol{i}t)| \geqslant 1.$$

Proof of the lemma. We can express log of the above quantity as

$$X := 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + \boldsymbol{i}t)| + \log|\zeta(\sigma + \boldsymbol{i}t)| = -\mathfrak{Re}\left(\sum_p 3\log(1 - p^{-\sigma}) + 4\log(1 - p^{-\sigma-\boldsymbol{i}t}) + \log(1 - p^{-\sigma-2\boldsymbol{i}t})\right).$$

Using the Taylor series expansion of log, we find

$$X = \sum_{n=1}^{\infty} c_n(3 + 4\cos\theta_n + \cos(2\theta_n)),$$

where $c_n = 1/m$ iff $n = p^m$ for a a prime p and $\theta_n = t\log n$.

Since $(3 + 4\cos\theta + \cos(2\theta)) = 2(1 + \cos\theta)^2$, we have $X \geqslant 0$. □