Lecture 2

Differentiable maps $f\colon \mathbb{R}^m \to \mathbb{R}^k$

In this section we overview the notions of continuity and differentiability for functions defined on open subsets of $\mathbb{R}^m$ with values in $\mathbb{R}^k$. You have studied these notions in the case $m = k = 1$ in the course of introductory calculus. In the Complex Variables course we will need it in a specific case $m = k = 2$. However, since all the definitions work equally well for any m and k, we will give general definitions.

For $v \in \mathbb{R}^n$ we denote by $|v|$ the length of the corresponding vector.

Definition 1 (Open sets)**.** A subset $U \subset \mathbb{R}^n$ is open if for any $x \in U$ there exists $\epsilon > 0$ such that the open ball of radius ϵ with its center at x also belongs to U:

$$B_\epsilon(x) := \{y \in \mathbb{R}^n \mid |x - y| < \epsilon\} \subset U.$$

From now on let $U \subset \mathbb{R}^m$ be an open set and consider a map $f\colon U \to \mathbb{R}^k$.

Definition 2 (Continuity)**.** Map $f\colon U \to \mathbb{R}^k$ is continuous at $x_0 \in U$ if

$$\lim_{h\to 0} f(x_0 + h) = f(x),$$

i.e., for any $\delta > 0$ there exists $\epsilon = \epsilon(\delta) > 0$ such that

$$f(U \cap B_\epsilon(x_0)) \subset B_\delta(f(x_0)).$$

Theorem 3. *A map $f\colon U \to \mathbb{R}^k$ is continuous at every point $x_0 \in U$ if and only if for any open $V \subset \mathbb{R}^k$, its preimage $f^{-1}(V) \subset U$ is also open.*

Proof. Exercise. □

Definition 4. Map $f\colon U \to \mathbb{R}^k$ is *differentiable* at $x_0 \in U$ if there exists a linear operator $A\colon \mathbb{R}^m \to \mathbb{R}^k$ (one can think of A as a $k \times m$ matrix) such that for $h \in \mathbb{R}^m$ in a small neighbourhood of 0

$$f(x_0 + h) = f(x_0) + A(h) + o(h),$$

where $o(h)$ is a *little o of h*, i.e,

$$\lim_{h\to 0} \frac{|o(h)|}{|h|} = 0.$$

Roughly speaking, map $f\colon U \to \mathbb{R}^k$ is differentiable at $x_0 \in U$ if the difference $f(x_0 + h) - f(x_0)$ as a function of h is 'well-approximated' by a linear map $h \mapsto A(h)$. Linear map A is called *derivative* or *differential* of f at x and is often denoted as $df_x \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^k)$.

Example 5. Consider a map $f\colon \mathbb{C} \to \mathbb{C}$, $f\colon z \mapsto z^2$. Identifying every complex number $a + \boldsymbol{i}b$ with a point (a, b) in $\mathbb{R}^2$, we can interpret f as a map between $\mathbb{R}^2$ and $\mathbb{R}^2$.

Claim: $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at any point $(a, b) \in \mathbb{R}^2$.

Proof: In coordinates, f is given by

$$f\colon \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a^2 - b^2 \\ 2ab \end{bmatrix}.$$

Near point $x = \begin{bmatrix} a \\ b \end{bmatrix}$ for $h = \begin{bmatrix} h_a \\ h_b \end{bmatrix}$ we have:

$$f\left(\begin{bmatrix} a + h_a \\ b + h_b \end{bmatrix}\right) - f\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2ah_a - 2bh_b \\ 2ah_b + 2bh_a \end{bmatrix} + \begin{bmatrix} h_a^2 - h_b^2 \\ h_a h_b \end{bmatrix} = \begin{bmatrix} 2a & -2b \\ 2b & 2a \end{bmatrix} \begin{bmatrix} h_a \\ h_b \end{bmatrix} + o(h).$$

Hence the differential at $\begin{bmatrix} a \\ b \end{bmatrix}$ is given by $df = \begin{bmatrix} 2a & -2b \\ 2b & 2a \end{bmatrix}$.

Remark 6. For any point (a, b), the matrix $\begin{bmatrix} 2a & -2b \\ 2b & 2a \end{bmatrix}$ belongs to the space M of matrices which we have introduced in the first lecture. As we will see further, this is not a coincidence!

Differentiable maps between spaces $\mathbb{R}^n$ satisfy all usual properties of differentiation.

- If $f, g\colon \mathbb{R}^m \to \mathbb{R}^k$ are differentiable at $x \in \mathbb{R}^k$, $\lambda \in \mathbb{R}$ then $f+g$ and $\lambda \cdot f$ are differentiable with $d(f+g)_x = df_x + dg_x$, $d(\lambda \cdot f)_x = \lambda \cdot df_x$
- If $f\colon \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $x \in \mathbb{R}^m$ and $g\colon \mathbb{R}^k \to \mathbb{R}^n$ is differentiable at $f(x)$, then $g \circ f\colon \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at x with
$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Isomorphism between $\mathbb{C}$ and M

Recall that $M := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

Consider any $w \in \mathbb{C}$ and define a map $m_w\colon \mathbb{C} \to \mathbb{C}$ which multiplies any complex number by w:
$$m_w\colon z \mapsto z \cdot w.$$

Identifying as usual $\mathbb{C}$ with $\mathbb{R}^2$ with coordinates $(\mathfrak{Re}\, z, \mathfrak{Im}\, z)$, we can see that m_w is a linear map between $\mathbb{R}^2$ and $\mathbb{R}^2$, and can be represented by 2×2 matrix. Let us call it A_w

Exercise 1. Check that the 2×2 matrix A_w representing m_w belongs to M. Show that the correspondence $w \mapsto A_w$ is indeed an isomorphism between $\mathbb{C}$ and M.

Remark 7. Geometrically, the map $m_w\colon \mathbb{R}^2 \to \mathbb{R}^2$ rotates the plane by the angle $\mathrm{Arg}(w)$ and scales it by a factor of $|w|$.

Holomorphic functions $f\colon \mathbb{C} \to \mathbb{C}$

If we identify $\mathbb{C}$ with $\mathbb{R}^2$ then we can apply all of the above to any function $f\colon \mathbb{C} \to \mathbb{C}$. In particular, we can speak of *continuous* functions $f(z)$. We could also us Definition 4 o define **real**-differentiable functions $f\colon \mathbb{C} \to \mathbb{C}$. This is **not** the notion which will be focusing on in this course. Instead we introduce a new concept of *complex derivative*.

Let $U \subset \mathbb{C}$ be an open subset.

Definition 8. Function $f\colon U \to \mathbb{C}$ is called *holomorphic* at $z_0 \in U$ if there exists a limit as h **approaches** 0 **in** $\mathbb{C}$:
$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}, \qquad h \in \mathbb{C}.$$

This complex number is called *complex derivative* of f at z_0:
$$f'(z_0) := \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Function $f(z)$ is called *holomorphic* if it is holomorphic at any point $z_0 \in U$.

Remark 9. If $f\colon U \to \mathbb{C}$ is complex differentiable at z_0, then $f\colon U \to \mathbb{C}$ is also real-differentiable (in the sense of Definition 4). Indeed, Definition 8 is equivalent to the identity:
$$f(z_0 + h) = f(z_0) + f'(z_0)h + o(h), \qquad h \in \mathbb{C}.$$

Hence the real differential df_{z_0} is represented by a map
$$m_{f'(z_0)}\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (a + \boldsymbol{i}b) \mapsto f'(z_0) \cdot (a + \boldsymbol{i}b).$$

As shows the following example, the converse is not true.

Example 10. Function $f(z) = \bar{z}$ is real-differentiable on its domain as a map $\mathbb{R}^2 \to \mathbb{R}^2$ but not holomorphic. Indeed, the limit
$$\frac{f(z_0 + h - f(z_0))}{h} = \frac{\bar{h}}{h}$$
does not exist, since the expression on the right hand side goes to 1 as h approaches 0 along the real axis: $h = t$, $t \to 0$, $t \in \mathbb{R}$ and to -1 along the imaginary axis: $h = \boldsymbol{i}t$, $t \to 0$, $t \in \mathbb{R}$.

Proposition 11. *If $f(z), g(z)$ are holomorphic at z_0, then $f+g, f\cdot g$ and f/g (provided $g(z_0)\neq 0$) are holomorphic at z_0 with the complex derivatives given by the usual formulas.*

If $f(z)$ is holomorphic at z_0 and $g(z)$ is holomorphic at $f(z_0)$, then $(g\circ f)$ is also holomorphic at z_0 with $(g\circ f)'(z_0) = g'(f(z_0))f'(z_0)$.

Proof. Exercise. □

Example 12. Proposition 11 implies that all polynomials $p(z)$ and rational functions $p(z)/q(z)$ are holomorphic on their domains.

For instance, $f(z) = z^2$ is holomorphic at any $z_0 = a+\boldsymbol{i}b$ with $f'(z_0) = 2z_0 = 2(a+\boldsymbol{i}b)$ (Compare with Example 5).

Cauchy-Riemann equations

Let $f(z)$ be a complex-valued function of a complex argument. Assume that it is differentiable as a map $\mathbb{R}^2\to\mathbb{R}^2$ at z_0. In this section, we derive necessary and sufficient conditions on partial derivatives of f for $f(z)$ to be holomorphic at $z_0 = x_0+\boldsymbol{i}y_0$.

Let $f(x+\boldsymbol{i}y) = u(x,y)+\boldsymbol{i}v(x,y)$ be the real and imaginary parts of $f(z)$.

Theorem 13 (CR equations)**.** *Differentiable mapping $f(x+\boldsymbol{i}y) = u(x,y)+\boldsymbol{i}v(x,y)$, $f\colon\mathbb{R}^2\to\mathbb{R}^2$ is holomorphic at $z_0 = x_0+\boldsymbol{i}y_0$ if and only if the following Cauchy-Riemann equations are satisfied:*

$$\boxed{\begin{cases}\dfrac{\partial u}{\partial x}(x_0,y_0) = \dfrac{\partial v}{\partial y}(x_0,y_0)\\[2mm] \dfrac{\partial v}{\partial x}(x_0,y_0) = -\dfrac{\partial u}{\partial y}(x_0,y_0)\end{cases}}$$

Proof. Necessity: Assume that $f(z)$ is holomorphic at z_0. Then the limits with $h\in\mathbb{R}$, $h\to 0$

$$\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$$
$$\lim_{h\to 0}\frac{f(z_0+\boldsymbol{i}h)-f(z_0)}{\boldsymbol{i}h}$$

must coincide. Equating separately the real and the imaginary parts of these limits we find:

$$\lim_{h\to 0}\frac{u(x_0+h)-u(x_0)}{h} = \lim_{h\to 0}\frac{v(y_0+h)-v(y_0)}{h}$$
$$\lim_{h\to 0}\frac{v(y_0+h)-v(y_0)}{h} = -\lim_{h\to 0}\frac{u(x_0+h)-u(x_0)}{h}$$

which is exactly the CR equations.

Sufficiency: Assume that the CR equations are satisfied. Then the differential of $f\colon\mathbb{R}\to\mathbb{R}$ at $z_0 = x_0+\boldsymbol{i}y_0$ is given by the Jacobi matrix

$$df_{(x_0,y_0)} = \begin{bmatrix}\dfrac{\partial u}{\partial x}(x_0,y_0) & \dfrac{\partial u}{\partial y}(x_0,y_0)\\[2mm] \dfrac{\partial v}{\partial x}(x_0,y_0) & \dfrac{\partial v}{\partial y}(x_0,y_0)\end{bmatrix}.$$

Under Cauchy-Riemann equations this matrix is of the form

$$\begin{bmatrix}a & -b\\ b & a\end{bmatrix},\ a,b\in\mathbb{R}.$$

For any matrix $A = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ the corresponding map between $\mathbb{R}^2\simeq\mathbb{C}$ and $\mathbb{R}^2\simeq\mathbb{C}$ is the multiplication by $a+\boldsymbol{i}b$ (see discussion about isomorphism between $\mathbb{C}$ and M above). Hence the complex derivative exists and is given by

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0)+\boldsymbol{i}\frac{\partial v}{\partial x}(x_0,y_0).$$

□

Using CR equations, we can alternatively write $f'(z_0)$ as follows

$$f' = \frac{\partial u}{\partial x} - \boldsymbol{i}\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \boldsymbol{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \boldsymbol{i}\frac{\partial u}{\partial y}.$$

Harmonic functions

Consider function $f(z)$ which is holomorphic in an open domain $U \subset \mathbb{C}$. Then we have CR equations:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$$

One of the fundamental theorems of complex analysis, which we prove later, implies that functions u and v must be infinitely differentiable. Assuming for now that u and v merely continuously differentiable. Using the fact that $\frac{\partial^2}{\partial x \partial y}$ and $\frac{\partial^2}{\partial y \partial x}$ coincide on continuously differentiable functions, we can compute:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v^2}{\partial x \partial y} = 0.$$

In other words, $\Delta u = 0$, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the *Laplace operator*. Similarly $\Delta v = 0$.

Definition 14. If a twice-differentiable function $u(x,y)$ satisfies the *Laplace equation*

$$\Delta u = 0$$

on an open region U, then we say that u is *harmonic* in U.

Example 15. Function $u(x,y) = x^2 - y^2$ is Harmonic in $\mathbb{R}^2$. Indeed, $u''_{xx} + u''_{yy} = 2 - 2 = 0$. Alternatively, we could have just observed that $u(x,y) = \mathfrak{Re}((x + \boldsymbol{i}y)^2)$ is the real part of a holomorphic function $f(z) = z^2$.

Harmonic functions and the Laplace equation play extremely important role in the modern mathematics. We will see that theories of harmonic functions on the plane and holomorphic function are intimately related.

Exercise 2. Find a degree 4 polynomial $P(x,y) \in \mathbb{R}[x,y]$ such that $\Delta P = 0$.