Lecture 20

Conformal mappings

Given a differentiable map $F\colon \mathbb{R}^2 \supset U \to \mathbb{R}^2$ we have its induced action on vectors $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$ at $\mathbf{x} \in U$ via

$$F_*(\mathbf{x})v = \begin{bmatrix} \frac{\partial F_1(\mathbf{x})}{\partial x_1} v_1 + \frac{\partial F_1(\mathbf{x})}{\partial x_2} v_2 \\ \frac{\partial F_2(\mathbf{x})}{\partial x_1} v_1 + \frac{\partial F_2(\mathbf{x})}{\partial x_2} v_2 \end{bmatrix} = \mathrm{Jac}(F(\mathbf{x})) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2,$$

where $\mathrm{Jac}(F(\mathbf{x}))$ is the Jacobi matrix of F at $\mathbf{x}$.

One of the key features of holomorphic mappings $f\colon U \to \mathbb{C}$ is *conformality*. Namely, given a point z_0 such that $f'(z_0) \neq 0$, map f preserves angles between rooted vectors at z_0.

Lemma 1. *Linear map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ preserves oriented angles between vectors if and only if A is of the form*

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$.

Proof. Exercise. □

Theorem 2. *Real differentiable map $f\colon U \to \mathbb{C}$ has complex derivative $f'(z_0) \neq 0$ at $z_0 \in U$ if and only if $f(z)$ preserves angles between vectors at z_0.*

Proof. By the Lemma, f preserves angles if and inly if its real derivative is given by a matrix of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

with $a^2 + b^2$. By Cauchy-Riemann identities a real differentiable map has differential of this form if and only if it is holomorphic. □

Definition 3. A holomorphic function $f\colon U \to \mathbb{C}$ is called a *conformal map*, if its derivative does not vanish.

Example 4. Function $f(z) = z^2$ is a conformal mapping from $\mathbb{C} - \{0\}$ onto $\mathbb{C} - \{0\}$.

Fundamental question of complex analysis is to classify open subsets $U \subset \mathbb{C}$ up to conformal equivalence. This raises two questions:

Question. *Given $U, V \subset \mathbb{C}$ does there exists a holomorphic bijection $f\colon U \to V$ (such f is called conformal equivalence between U and V)?*

Question. *Given $U \subset \mathbb{C}$ what is the group of holomorphic automorphisms $f\colon U \to U$?*

In general, these questions are difficult to answer. However, both of them have a remarkably simple answer if we additionally assume that U and V are simply connected. This is the content of the celebrated Riemann mapping theorem.

Riemann mapping theorem

Theorem 5 (Riemann mapping theorem)**.** *Suppose that a connected open subset $U \subsetneq \mathbb{C}$ is proper and simply-connected. Given $z_0 \in U$ there exists a unique bijective holomorphic function $F\colon U \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) \in \mathbb{R}_{>0}$.*

Proof of the uniqueness. If F_1 and F_2 are two such functions, then $H := F_1 \circ F_2^{-1}$ is an automorphism of $\mathbb{D}$ fixing the origin. By a consequence of Schwarz lemma, any such automorphism has a form $H\colon z \mapsto e^{i\theta} z$. Conditions $F_1'(z_0), F_2'(z_0) \in \mathbb{R}_{>0}$ imply that $H'(0)$ is also real and positive, therefore $H(z) = z$. □

The existence part of the Riemann mapping theorem is one of the most important and fundamental theorems of our course. Surprisingly, for a result of such significance, this theorem has a rather easy proof.

Normal families

Definition 6. Consider an open connected subset $U \subset \mathbb{C}$ and a family $\mathcal{F}$ of complex-valued functions on U. We say that $\mathcal{F}$ is

- *normal* if for every sequence $\{f_i\}$ from $\mathcal{F}$ there is a subsequence $\{f_{k_i}\}$ converging on compact subsets of U.
- *locally uniformly bounded* if for any compact $K \subset U$ there exists $M > 0$ such that $|f(z)| < M$ for all $f \in \mathcal{F}$ and $z \in K$.
- *equicontinuous* on a compact set $K \subset U$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\text{for all } (z, w) \in K \text{ with } |z - w| < \delta \text{ and any } f \in \mathcal{F}$$

$$|f(z) - f(w)| < \epsilon.$$

Remark 7. If there exists a constant $M > 0$ such that for every $f \in \mathcal{F}$ and $z \in K$ we have $|f'(z)| < M$, then $\mathcal{F}$ is equicontinuous on K.

Theorem 8 (Montel's theorem)**.** *Suppose $\mathcal{F}$ is a family of holomorphic functions on U that is uniformly bounded on compact subsets of U. Then*

1. *$\mathcal{F}$ is equicontinuous on every compact subset $K \subset U$;*
2. *$\mathcal{F}$ is a normal family.*

Proof. First, let us prove equicontinuity. This part relies in Cauchy's theorem and essentially uses the fact that the functions in our family are holomorphic.

Given $K \subset U$ take $r > 0$ such that for every $z_0 \in K$ we have $B_{3r}(z_0) \subset U$. Let $N_{2r}(K)$ be a $2r$-neighbourhood of K:

$$N_{2r}(K) = \{z \in \mathbb{C} \mid |z - w| \leqslant 2r \text{ for some } w \in K\}.$$

By our assumption on r, $N_{2r}(K)$ is compact and contained in U. Hence, $\mathcal{F}$ is uniformly bounded by some constant $B > 0$ on $N_{2r}(K)$.

Given any $z, w \in K$ inside a disk $B_r(z_0)$ for $\gamma = \partial B_{2r}(z_0)$, we have

$$f(z) - f(w) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma f(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right] d\zeta.$$

Now, for any $\zeta \in \gamma$

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = |z - w| \left| \frac{1}{(\zeta - z)(\zeta - w)} \right| \leqslant \frac{|z - w|}{r^2}.$$

Therefore

$$|f(z)-f(w)| \leqslant \frac{B}{r}|z-w|.$$

Since this inequality holds for all $f \in \mathcal{F}$ and $z, w \in K$, we conclude that $\mathcal{F}$ is equicontinuous on K.

Now we prove that $\mathcal{F}$ is normal under assumptions of equicontinuity and uniform boundedness. This a version of a general statement known as *Arzelà–Ascoli theorem*.

Let $\{f_n\}$ be a sequence from $\mathcal{F}$. Pick an everywhere dense sequence of points $\{w_j\} \subset U$, e.g., take all the points with rational coordinates. The sequence of values $\{f_n(w_1)\}_{n\in\mathbb{N}}$ is bounded, therefore we can choose a convergent subsequence $\{f_{n,1}(w_1)\}_{n\in\mathbb{N}}$. At the next step we consider a bounded sequence $\{f_{n,1}(w_2)\}$ and choose its convergent subsequence $\{f_{n,2}(w_2)\}$. Repeating this process for all points w_i, we extract a *diagonal* sequence of functions

$$\{g_n\}_{n\in\mathbb{N}}, \qquad g_n := f_{n,n}$$

such that the values at each of the points w_i converge.

We claim that $\{g_n\}$ uniformly converges on K. Fix $\epsilon > 0$. Since $\{g_n\}$ is uniformly equicontinuous on K, we can find the corresponding $\delta > 0$ and cover K by a finite collection of balls $B_\delta(w_1), d \dots B_\delta(w_k)$.

For $n, m > N(\epsilon)$ large enough, and $z \in B_\delta(w_j)$ we have

$$|g_n(z)-g_m(z)| \leqslant |g_n(z)-g_n(w_j)| + |g_n(w_j)-g_m(w_j)| + |g_m(w_j)-g_m(z)| < 3\epsilon,$$

where the first and the last summands are bounded due to equicontinuity, and the middle term due to the fact that $\{g_n(w_j)\}_{n\in\mathbb{N}}$ converges for $j \in 1, \dots k$. □

Proof of the Riemann mapping

Lemma 9. *If $\{f_n\}$ is a sequence of injective holomorphic functions in $U \subset \mathbb{C}$ that converges uniformly on every compact subset $K \subset U$ to a holomorphic function $f(z)$, then $f(z)$ is either a constant or also injective.*

Proof. Assume on the contrary that $f(z_1) = f(z_2)$ and f is not constant. Then the function $g(z) := f(z) - f(z_1)$ has isolated zeros at z_1 and z_2, while $g_n(z) := g(z) - g(z_1)$ have isolated zero only at z_1.

By argument principle for a small circle $C_r(z_2)$ enclosing z_2, we have

$$\frac{1}{2\pi \boldsymbol{i}} \int_{C_r(z_2)} \frac{g'(z)}{g(z)} dz = 1,$$

while

$$\frac{1}{2\pi \boldsymbol{i}} \int_{C_r(z_2)} \frac{g_n'(z)}{g_n(z)} dz = 0.$$

This is a contradiction, since the integrands are uniformly convergent on $C_r(z_2)$. □

Now we turn onto the proof of the theorem.

Step 1. Suppose $U \subsetneq \mathbb{C}$ is a proper simply-connected subset of $\mathbb{C}$. We claim that U is conformally equivalent to an open subset of $\mathbb{D}$.

Indeed, pick $\alpha \notin U$. There exists a well defined function $f(z) = \sqrt{z-\alpha}$ in U. Clearly $f(z)$ does not take the same value twice, nor the opposite values, since $f(z)^2 = z - \alpha$.

Now, by open mapping, $f(U)$ contains some open disk $D = B_r(w)$, therefore it misses the open disk $B_r(-w)$. Therefore the map $F(z) = \frac{1}{f(z)+w}$ maps bijectively U onto an open subset of $B_{1/r}(0)$.

Step 2. Composing with scalings and translations if necessary, by the first step, we may assume that $U \subset \mathbb{D}$ and $0 \in U$. So we have a non-empty family

$$\mathcal{F} := \{f : U \to \mathbb{D} \mid \text{holomorphic, injective and } f(0) = 0\}.$$

Clearly $\mathcal{F}$ is uniformly bounded. Let

$$s := \sup_{f \in \mathcal{F}} |f'(0)|.$$

This supremum is finite, since $|f'(0)|$ is bounded by Cauchy's estimates. Choose a sequence $\{f_n\} \subset \mathcal{F}$ such that $|f_n'(0)| \to s$. By Montel's theorem, sequence $\{f_n\}$ converges to a holomorphic function $f(z)$, moreover, by the above Lemma, since $f(z)$ is not constant[1], we see that $f(z)$ is injective. Also, by continuity $|f(z)| \leqslant 1$ on U, so $f \in \mathcal{F}$ and $|f'(0)| = s$.

Step 3. Function $f(z)$ constructed above conformally maps U onto $\mathbb{D}$.

Suppose, on the contrary that $\alpha \in \mathbb{D}$ does not belong to $f(U)$. Consider an automorphism of $\mathbb{D}$ interchanging α and 0

$$\psi_\alpha := \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Then simply connected region $W := (\psi_\alpha \circ f)(U)$ does not contain 0 and we can define $g(w) := \sqrt{w}$ in W. Consider new function

$$F := \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f.$$

It is easy to see that $F(0) = 0$, and $F : U \to \mathbb{D}$ is injective, so $F \in \mathcal{F}$. Then we have:

$$f = \Phi \circ F,$$

where $\Phi : \mathbb{D} \to \mathbb{D}$ with $\Phi(0) = 0$ is non-injective. By Schwarz lemma we must have $|\Phi'(0)| < 1$, since otherwise Φ is an automorphism. Therefore

$$|f'(0)| < |F'(0)|$$

which contradicts the maximality of $|f'(0)|$.

This proves that $f : U \to \mathbb{D}$ is not only injective but also surjective, and f^{-1} is well-defined, since $f' \neq 0$ on U.

[1] By the way, why?