Lecture 21

Riemann mapping theorem

Examples of conformal equivalences

Once we have proved Riemann mapping theorem, let us consider introduce explicit conformal equivalences between various regions.

Example 1. Let $\mathbb{H}$ be the upper half-plane.

$$\mathbb{H} := \{\mathfrak{Im}(z) > 0\}.$$

Cayley transform $F(z) := \frac{\boldsymbol{i}-z}{\boldsymbol{i}+z}$ and its inverse $G(w) := \boldsymbol{i}\frac{1-w}{1+w}$ provide equivalences between $\mathbb{H}$ and $\mathbb{D}$:

$$F\colon \mathbb{H} \to \mathbb{D}; \quad G\colon \mathbb{D} \to \mathbb{H}.$$

Example 2. For a fixed $\alpha \in (0, 2]$ let $f(z) = z^\alpha$ be the principle branch defined in $\mathbb{H}$. Then $f(\mathbb{H})$ is an infinite sector between the rays $[0, +\infty)$ and $e^{\boldsymbol{i}\pi\alpha}[0, +\infty)$.

We can equivalently rewrite

$$f(z) = \alpha \int_0^z \zeta^{-\beta} d\zeta$$

where $\beta = 1 - \alpha$. This representation will be significant in the view of the Schwarz-Christoffel integrals below.

Example 3. The map $z \mapsto \log z$ defined as the principle branch of logarithm takes the upper half-plane to the strip

$$S := \{u + \boldsymbol{i}v \mid u \in \mathbb{R}, v \in (0, \pi)\}$$

and the upper half-disk to a half-strip.

Example 4. Function $\sin(z)$ is a composition of $e^{\boldsymbol{i}z}$, $\boldsymbol{i}z$ and $-1/2(z+1/z)$ taking half-strip $\{\mathfrak{Re}(z) \in (-\pi/2, \pi/2), \mathfrak{Im}(z) > 0\}$ to the upper half-plane.

Applications to Dirichlet problem

Dirichlet problem in an open set $U \subsetneq \mathbb{C}$ with a *nice* (e.g., given by a simple closed piecewise smooth curve γ) boundary ∂U consists of solving equation for $u\colon \overline{U} \to \mathbb{R}$

$$\begin{cases} \Delta u = 0 & \text{in } U \\ u = f & \text{on } \partial U, \end{cases} \tag{1}$$

where $f\colon \partial U \to \mathbb{R}$ is a given function. Normally, the second equation is imposed only at points of continuity of f.

We have seen that if $U = \mathbb{D}$ is the unit disk and f is piecewise continuous, solution to (1) is given by the Poisson integral:

$$u(w) = P_f(w) := \frac{1}{2\pi} \int_{|z|=1} K_w(e^{\boldsymbol{i}\alpha}) f(e^{\boldsymbol{i}\alpha}) d\alpha,$$

where $K_w(z) := \mathfrak{Re}\left(\frac{z+w}{z-w}\right)$ is the Poisson kernel.

Now, assume we want to solve Dirichlet problem in an infinite strip

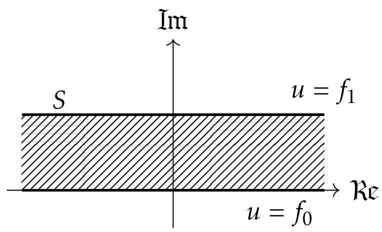
$$S := \{0 < \mathfrak{Im}(z) < 1\}.$$

Then we have two mappings $F\colon \mathbb{D} \to S$ and $G\colon S \to \mathbb{D}$

$$F(w) = \frac{1}{\pi}\log\left(\boldsymbol{i}\frac{1-w}{1+w}\right) \quad \text{and} \quad G(z) = \frac{\boldsymbol{i} - e^{\pi z}}{\boldsymbol{i} + e^{\pi z}}.$$

If $u(w)$ is a harmonic function in $\mathbb{D}$, then $u(G(z))$ will be a harmonic function in S. Mapping G extend to a continuous map defined on $\overline{S}$ sending line $\mathfrak{Im} = 1$ to the lower half-circle and line $\mathfrak{Im} = 0$ to the upper half-circle.

If we want to solve Dirichlet problem in S we can now translate it to a problem in $\mathbb{D}$.



Given continuous **bounded** real-valued functions f_0 and f_1 we use F to pull-back them to functions on $\partial\mathbb{D} - \{\pm 1\}$.

$$\tilde{f}(e^{\boldsymbol{i}\varphi}) = f_1(F(e^{\boldsymbol{i}\varphi}) - \boldsymbol{i}), \quad -\pi < \varphi < 0$$

$$\tilde{f}(e^{\boldsymbol{i}\varphi}) = f_0(F(e^{\boldsymbol{i}\varphi})), \quad 0 < \varphi < \pi.$$

Then we can solve the corresponding Dirichlet problem in $\mathbb{D}$

$$\tilde{u}(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_w(e^{\boldsymbol{i}\alpha})\tilde{f}(e^{\boldsymbol{i}\alpha})d\alpha.$$

Then $u(z) := \tilde{u}(G(z))$ is the desired function. Changing the integration variable and working through the necessary algebra, we can arrive at an explicit formula

$$u(x,y) = \frac{\sin \pi y}{2}\left(\int_{\mathbb{R}} \frac{f_0(x-t)}{\cosh \pi t - \cos \pi y}dt + \int_{\mathbb{R}} \frac{f_1(x-t)}{\cosh \pi t + \cos \pi y}dt\right).$$

Remark 5. Of course the same scheme works as long as we can consider a conformal equivalence $f : U \to \mathbb{D}$ and extend it continuously to the boundary.

Behavior at the boundary

For applications of the Riemann mapping theorem, it is important to understand when conformal equivalence $f : U \to V$ continuously extends to the boundary $f : \overline{U} \to \overline{V}$. The following example shows that in general this is not the case.

Example 6. Map $z \mapsto \sqrt{z^2 - 1}$ from $\{\mathfrak{Re}(z) > 0\} - (0, 1] \to \{\mathfrak{Re}(z) > 0\}$ does not extend to a continuous map on $\{\mathfrak{Re}(z) \geqslant 0\}$.

Luckily, such extension exists for many regions which we face in practice. The following theorem gives the statement for *polygonal regions* P, namely bounded simply-connected open sets whose boundary is a polygonal line p.

Theorem 7. *If $F\colon \mathbb{D} \to P$ is a conformal map, then F extends to a continuous bijection from $\overline{D}$ to $\overline{P}$.*

We will not prove the above theorem (if you are interested, you can find a proof in Ch. 8 of Stein-Shakarchi).

Remark 8. The above theorem remain valid if we consider bounded simply-connected open sets with simple piecewise smooth boundary.

It turns out, that in the case of polygonal regions, the conformal equivalence $F\colon \mathbb{H} \to P$ can be found explicitly.

Theorem 9. *A holomorphic function giving equivalence between* $\mathbb{H}$ *and* P*, where* P *is a polygon with angles* $\{\pi\alpha_k\}$ *is given by*

$$F(z) = c_1 S(z) + c_2,$$

where $S(z)$ *is the Schwarz-Christoffel integral*

$$S(z) := \int_0^z \frac{d\zeta}{(\zeta - A_1)^{\beta_1} \dots (\zeta - A_n)^{\beta_n}}.$$

Here $A_1 < A_2 < \dots < A_n$ *are* n *distinct points on the real axis and* $\beta_k = 1 - \alpha_k$*.*

Remark 10. Since $\sum \beta_k = 2$, the Schwarz-Christoffel integral converges at ∞, and by Cauchy theorem there exists a finite limit $\lim_{z\to\infty} S(z)$.

Example 11. *Elliptic integral*

$$\kappa(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^{1/2}(1-k^2\zeta^2)^{1/2}}, \quad k \in (0,1)$$

maps upper half-plane onto rectangle with vertices $\pm K, \pm K + \boldsymbol{i}K'$.