Lecture 22

Analytic continuation

We have encountered in this course that often a given holomorphic function $f: U \to \mathbb{C}$ can extended to a holomorphic function in a larger domain. By *rigidity* of holomorphic functions it follows that if such extension exists, then it must be unique. Today we will discuss some necessary examples ensuring the existence of such extension, called *analytic continuation*.

Singular points on ∂U

Definition 1. Let $f: U \to \mathbb{C}$ be a holomorphic function, and consider a point $p \in \partial U$ on the boundary of U. Point p is a *regular point* of f, if there exists a holomorphic function $g(z)$ defined in a small disk $B_\epsilon(p)$, such that on $U \cap B_\epsilon(p)$ functions $f(z)$ and $g(z)$ coincide.

Otherwise, we call p a *singular point*.

Existence of singular points is the only obstruction to the extension of a holomorphic function in a disk $f: B_r(z_0) \to \mathbb{C}$ to a larger disk $B_R(z_0)$.

Theorem 2. *Suppose $f(z)$ is analytic in $B_r(z_0)$ and its power series has radius of convergence $0 < r < \infty$. Then $f(z)$ has at least one singular point $p \in \partial B_r(z_0)$.*

Proof. Assume on the contrary that all points $p \in \partial B_r(z_0)$ are regular. By compactness, we can cover $\partial B_r(z_0)$ with a finite number of open disks B_i and find a holomorphic function $g_i(z)$ in each B_i such that $g_i(z) = f(z)$ in $B_r(z_0) \cap B_i$.

We claim that $\{f\} \cup \{g_i\}_i$ yields a well-defined function $F(z)$ on $U := B_r(z_0) \cup_i B_i$. For $z \in U$ we need to check that $f(z)$ is ambiguously defined. We can assume that $z \in B_i \cap B_j$.

By our assumption, functions $f(z)$, $g_i(z)$ and $g_j(z)$ coincide on $B_i \cap B_j \cap B_r(z_0)$. Therefore, $g_i(z)$ and $g_j(z)$ are two analytic continuations of the function $f(z) = g_i(z) = g_j(z)$ defined on $B_i \cap B_j \cap B_r(z_0)$. Since $B_i \cap B_j$ is connected, we have $g_i(z) = g_j(z)$ by the uniqueness of analytic continuation.

We have constructed a holomorphic function (by abuse of notation we call it $f(z)$) on the open set $U = B_r(z_0) \cup_i B_i \supset \overline{B}_r(z_0)$. Therefore $f(z)$ is holomorphic in some larger open disk $B_R(z_0)$, and has radius of convergence $\geqslant R > r$. Contradiction. □

There exist holomorphic functions in $B_r(z_0)$ such that all points on $\partial B_r(z_0)$ are singular, and the function cannot be extended holomorphically to any larger set.

Example 3. Consider $f(z) = \sum_{n=0}^{\infty} z^{2^n}$. Clearly $f(z)$ is holomorphic in $B_1(0)$ and unbounded in a neighbourhood of 1.

Since $f(z)$ satisfies functional equation

$$f(z^2) = f(z) - z,$$

we have that $f(z)$ is also unbounded at -1. Repeating this argument, we find that all points of the form $\zeta = e^{2\pi m/2^n}$ are singular. Since singular set is closed, it must coincide with the whole unit circle.

In the previous example $f(z)$ had singular points because it was unbounded. The following example shows that $f(z)$ can be even continuous up to $\partial B_1(0)$, yet all points on

Exercise 1. Fix $\alpha > 0$. Then function

$$f(z) = \sum 2^{-n\alpha} z^{2^n}$$

is holomorphic in $B_1(0)$, extends to a continuous function on $\overline{B}_1(0)$, yet it does not extend to a holomorphic function in any larger open set.

Hint: function $u(t) := f(e^{2\pi i t})$ is continuous nowhere differentiable.

Function elements and germs

To formulate main results about analytic continuation in a clean concise way, we will need to introduce several new notions.

Definition 4. A *function element* is a pair (f, U), where f is holomorphic function in U. We say that two function elements (f, U) and (g, W) are *direct continuations* of each other if

1. $U \cap W$ is nonempty and connected;
2. $f(z) = g(z)$ on $U \cap W$.

We will write $(f, U) \sim (g, W)$.

Remark 5. The crucial feature of complex analysis is that the notion of direct continuation is **not transitive!** Namely, we might have $(f_1, U_1) \sim (f_2, U_2) \sim \cdots \sim (f_k, U_k)$, with $U_1 \cap U_k$ connected, yet $f_1 \neq f_k$ on $U_1 \cap U_k$.

Example 6. Consider $U_k = \{\arg(z) \in (\frac{\pi k}{2}, \frac{\pi k}{2} + \pi)\}$, $k = 0, 1, 2, 3$. In each U_k we can define unambiguously the function $f_k(z) = |z|^{1/2} e^{i \arg(z)/2}$, where the branch of $\arg(z)$ is chosen according to definition of U_k.

Then we have $(f_0, U_0) \sim (f_1, U_1) \sim (f_2, U_2) \sim (f_3, U_3)$, yet for $z_0 = (2 + i)^2 = (3 + 4i) \in U_1 \cap U_3$ we have $f_1(z_0) = 2 + i$ and $f_3(z_0) = -2 - i$.

Definition 7. A *germ* $[f]_\zeta$ of a holomorphic function f defined in a neighbourhood of ζ is the equivalence class of function elements (f, U) with $\zeta \in U$ where two function elements (f_1, U_1) and (f_2, U_2) are equivalent, if $f_1 = f_2$ on $U_1 \cap U_2$.

Remark 8. Definition of germ makes sense essentially for any functional space on a topological space, e.g., one can define germs of smooth functions on a smooth manifold, algebraic function on an algebraic variety etc. In our case, germ of a holomorphic function is uniquely defined by its Taylor's series.

Definition 9. *Analytic continuation* of a germ $[f]_{\zeta(0)}$ along a curve $\gamma = \zeta(t)$ between points $\zeta(0)$ and $\zeta(1)$ is a family of germs $[f]_{\zeta(t)}$ such that for any $t_0 \in [0, 1]$ there exists a functional element

$$(g, U)$$

representing $[f]_{\zeta(t)}$ for all t in a small neighbourhood of t_0.

Proposition 10. *Analytic continuation of a germ $[f]_\zeta$ along a given curve is unique, if exists.*

Proof. If $[f]_{\zeta(t)}$ and $[g]_{\zeta(t)}$ are two continuations along the same curve, then the set of $t \in [0, 1]$ such that $[f]_{\zeta(t)} = [f]_{\zeta(t)}$ is nonempty, closed and open. Therefore $[f]_{\zeta(t)} = [g]_{\zeta(t)}$ for all $t \in [0, 1]$. □

Monodromy theorem

Definition 11. Consider an open set U and a germ $[f]_\zeta$, $\zeta \in U$. We say that $[f]_\zeta$ admits an *unrestricted continuation* in U if $[f]_\zeta$ can be analytically continued along any curve in U.

Example 12. The germ defined by the principle branch of $\sqrt{z}$ admits unrestricted continuation in $\mathbb{C}^* := \mathbb{C} - \{0\}$.

Theorem 13 (Monodromy theorem)**.** *Assume that the germ $[f]_\zeta$ admits an unrestricted continuation in U. If $\zeta_0(t)$ and $\zeta_1(t)$ are two homotopic curves from $\zeta(0)$ to $\zeta(1)$ then continuations along ζ_0 and ζ_1 at the endpoint $\zeta(1)$ coincide.*

Corollary 14. *If the germ $[f]_\zeta$ admits an unrestricted continuation in a **simply-connected** open set U, then there exists a holomorphic $g\colon U \to \mathbb{C}$ such that function element (g, U) represents the germ $[f]_\zeta$.*

Proof of the monodromy theorem. Let $\zeta_s(t)$, $s \in [0,1]$ be a homotopy between $\zeta_0(t)$ and $\zeta_1(t)$. Take $s_0 \in [0,1]$ and let $[f_{s_0}]_{\zeta(1)}$ be the result of analytic continuation of $[f]_{\zeta(0)}$ along ζ_{s_0}.

We claim that the set

$$\mathcal{S}(s_0) := \{s \mid [f_s]_{\zeta(1)} = [f_{s_0}]_{\zeta(1)}\},$$

of paths ζ_s such that the continuations along ζ_s and ζ_{s_0} coincide, is open.

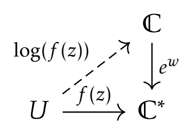
Indeed, cover the curve $\zeta_{s_0}(t)$ with small open disks $\{D_\alpha\}$ such that each germ $[f]_{\zeta_{s_0}(t)}$, $t \in [0,1]$ is represented by some (f_α, D_α). By compactness we can assume that there are finitely many disks. Then the same collection (f_α, D_α) defines analytic continuation along $\zeta_{s_\epsilon}(t)$ for $s_\epsilon \in (s_0 - \epsilon, s_0 + \epsilon)$ for some $\epsilon > 0$. This proves openness.

Since $\cup_{s_0} \mathcal{S}(s_0) = [0,1]$ and [0,1] is connected, we have that all possible continuations $[f_s]_{\zeta(1)}$ coincide. □

Example 15. Consider a holomorphic function $f\colon U \to \mathbb{C}$ in a simply-connected region such that $f \neq 0$ in U. Fix any $z_0 \in U$. In a neighborhood of a point $f(z_0)$ we can take a branch of logarithm and define locally in a neighbourhood of z_0 a function

$$G(z) := \log(f(z)).$$

We claim that function $G(z)$ admits unrestricted continuation on U. Let $\gamma(t)$ be any curve in U starting at z_0 and consider curve $f(\gamma(t))$ in $\mathbb{C}$.



Since $e^w\colon \mathbb{C} \to \mathbb{C}^*$ is a local bijection, we can lift a neighbourhood of any small arc in $\mathbb{C}^*$ to $\mathbb{C}$. This would define a function element which gives an analytic continuation along the arc.

Monodromy theorem ensures that there exists a holomorphic function extending $G(z) = \log(f(z))$ in U.

Picard's little theorem

Monodromy theorem is an important ingredient in one of the proofs of Picard's little theorem.

Theorem 16 (Picard's little theorem)**.** *A non-constant entire function $f(z)\colon \mathbb{C} \to \mathbb{C}$ misses at most one value.*

For the sake of contradiction, assume that $f(z)$ misses points $a, b \in \mathbb{C}$. By considering $(f(z) - b)/(a - b)$, we can assume that $f(z)$ misses 0 and 1. Theory of elliptic curves and modular functions implies the existence of a special λ-function

$$\lambda\colon \mathbb{H} \to \mathbb{C} - \{0,1\}.$$

such that $\lambda' \neq 0$. In particular, $\lambda\colon \mathbb{H} \to \mathbb{C} - \{0,1\}$ is a local bijection.

Remark 17. Topologically, map λ realizes *universal cover* of $\mathbb{C} - \{0,1\}$. Since fundamental group of $\mathbb{C} - \{0,1\}$ is $\pi_1(\mathbb{C} - \{0,1\}) \simeq F_2$ the free group with two generators, there is an action of F_2 on $\mathbb{H}$ and λ is constant on orbits of this action.

We can repeat the argument in the above example of logarithm and lift $f\colon \mathbb{C} \to \mathbb{C} - \{0,1\}$ to a function $\lambda^{-1}(f(z))$:

$$\begin{array}{ccc} & \overset{\lambda^{-1}(f(z))}{\dashrightarrow} & \mathbb{H} \\ & & \downarrow \lambda \\ \mathbb{C} & \xrightarrow{f(z)} & \mathbb{C} - \{0,1\}. \end{array}$$

Thus we would get an entire function with values in $\mathbb{H}$. Since $\mathbb{H}$ is conformally equivalent to $\mathbb{D}$, Liouville's theorem implies that $\lambda^{-1}(f(z))$ is constant. Contradiction.