Lecture 23

Applications of conformal mappings

Harmonic functions on two-dimensional domains occur in many physical and engineering problems. Complex analysis provides a nice an clean way allowing to solve many of these problems explicitly.

Steady heat flow

Fourier's law of heat conduction states that the heat flux $\mathbf{q}$ in a homogeneous isotropic medium U of constant thermal conductivity k is given by

$$\mathbf{q} = -k\nabla u$$

where $u = u(x, y)$ is the temperature in U. If temperature is constant in time, for any region $\Omega \subsetneq U$ bounded by curve γ, the flow of $\mathbf{q}$ through γ must be zero due to the energy conservation laws:

$$\int_\gamma \mathbf{q} \cdot d\mathbf{n} = 0$$

By Stokes theorem this is equivalent to

$$\int_\Omega \mathrm{div}\mathbf{q}\, dxdy = 0$$

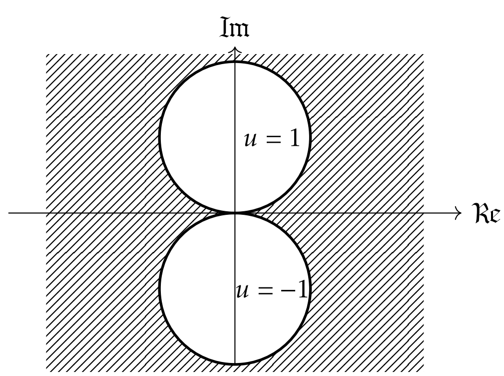
Since Ω is arbitrary, we get

$$\mathrm{div}\mathbf{q} = \Delta u = 0$$

so that $u(x, y)$ must be a harmonic function.

When $u = u(x, y)$ is specified on the boundary of U, we get Laplace's equation with Dirichlet boundary conditions.

Example 1. Let us compute the temperature u in the exterior of two disks $|z - \boldsymbol{i}| < 1$ and $|z + \boldsymbol{i}| < 1$, with Dirichlet conditions $u = 1$ on the first circle, and $u = -1$ on the second circle, and condition at infinity: $u \to 0$ as $z \to \infty$.



Notice that map $w = 1/z$ transforms the region outside two disks into a strip $\mathfrak{Im}(w) \in [-1/2; 1/2]$. Hence, we need to solve the problem

$$\begin{cases} \Delta U(\xi, \eta) = 0 \\ U(\xi, -1/2) = 1 \\ U(\xi, 1/2) = -1 \\ U \to 0 \quad \text{as } w = \xi + \boldsymbol{i}\eta \to 0 \end{cases}$$

There is a clear solution to this problem $U = -2\eta$. Therefore the required function is

$$u(x, y) = -2\eta = \frac{2y}{x^2 + y^2}.$$

Remark 2. Without the condition on the behavior at infinity, the problem fails to have a unique solution. For example $-2\eta + \Re\mathfrak{e}(e^{\pi w})$ is harmonic and satisfies the same boundary conditions.

Electrostatic potential

In electrostatics in $\mathbb{R}^3$ the electric field $\mathbf{E}$ is a gradient vector field of a certain potential function $u(x,y,z)$

$$\mathbf{E} = \langle u'_x, u'_y, u'_z \rangle.$$

Moreover, Gauss' law of electrostatics states that for a region $U \subset \mathbb{R}^3$ enclosed by surface Σ we have the identity for the surface integral

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{n} = \frac{Q}{\epsilon_0},$$

where Q is the total charge in U and ϵ_0 is the *electric constant*. In particular,

$$\int_\Sigma \mathbf{E} \cdot d\mathbf{n} = 0,$$

if U is charge-free.

If we assume that potential is independent of the third coordinate, and consider cylindrical regions $U \in \mathbb{R}^3$, we arrive at the 2-dimensional Gauss law

$$\int_\gamma \mathbf{E} \cdot d\mathbf{n} = 0$$

for any curve $\gamma \subset \mathbb{R}^2$ not enclosing any particles. As before, this implies

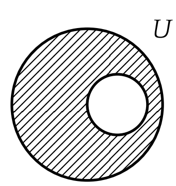
$$\text{div}\mathbf{E} = \Delta u = 0$$

Now, let $v(x,y)$ be a harmonic conjugate of $u(x,y)$ which always exists at least locally. Level curves of u and v are perpendicular to each other (Why?), moreover vector field $\mathbf{E}$ being the gradient of u is perpendicular to the level curves of u (which are called *equipotentials*), therefore $\mathbf{E}$ must be tangent to the level curves of v.

Remark 3. Notably, we can interchange the roles of u and v and consider electrostatic picture given by potential v. In this case the level curves of u become the field lines.

Example 4. Let us consider the situation in which one wants to compute the electrostatic potential in the vacuum region between an interior infinitely long cylinder and an exterior infinitely long cylindrical shell. This setup can be viewed as an idealization of a coaxial cable.

The problem as stated is essentially 2D in nature, so we will view it as a small disk inside a larger circle. We consider the general case in which the inner disk is off center: say the disk $|z - 2/5| \leqslant 2/5$ inside the circle $|z| = 1$.



Potential u solves the equation $\Delta u = 0$ in the region $U = \{|z| < 1\} - \{|z - 2/5| \leqslant 2/5\}$ with boundary conditions a and b on outer and inner circles respectively.

The idea is to map U to a simpler region in which we can solve the problem explicitly. The difficulty here is that U is not simply connected and Riemann mapping theorem does not apply.

However, we still can map U into rotationally symmetric annulus. Namely, consider automorphism of the unit disk

$$S_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha} z}$$

and let us find α such that $S_\alpha(z)$ moves circle $\{|z-2/5|<2/5\}$ to a circle centered at the origin.

We assume $\alpha \in \mathbb{R}$ and choose it in such a way that

$$\begin{cases} S_\alpha(0) = -R \\ S_\alpha(4/5) = R \end{cases}$$

Solving $S_\alpha(0) = -S_\alpha(4/5)$ for $\alpha \in (-1,1)$. we find $\alpha = 1/2$ so that

$$S(z) = \frac{2z-1}{2-z}.$$

which maps U to the annulus

$$\Omega = \{1/2 < |w| < 1\}$$

In this coordinates, we are looking for a function $\Phi(w)$ which solves Laplace equation in Ω with boundary values a and b. By the maximum principle, such solution is unique, therefore it must by rotationally invariant, as the boundary conditions are.

Claim. Any rotationally symmetric solution to the equation $\Delta\Phi(w) = 0$ must be of the form

$$\Phi(w) = A\ln|w| + B, A, B \in \mathbb{R}.$$

Therefore fitting constants A and B we find

$$\Phi(w) = \frac{a-b}{\ln 2}\ln|w| + a = (a-b)\log_2|w| + a$$

so that

$$\varphi(z) = (a-b)\log_2|S_\alpha(z)| + a = (a-b)\log_2\frac{(2x-1)^2+4y^2}{(x-2)^2+y^2} + a.$$

Remark 5. There is a version of Riemann mapping theorem which states that any connected region $U \subset \mathbb{C}$ with “one hole” is conformally equivalent to an annulus $\{r < |z| < R\}$. Moreover, two annuli are conformally equivalent if and only if they have the same ratio of radii R/r.

Potential flow

Consider incompressible fluid flow in a plane with velocity field $\mathbf{v}$. Assume that $\mathbf{v}$ is stationary, i.e., does not change in time.

The condition that $\mathbf{v}$ is incompressible is expressed in a familiar equation

$$\mathrm{div}\mathbf{v} = 0$$

which is satisfied if and only if locally $\mathbf{v}$ is given by

$$\mathbf{v} = \langle v_1, v_2\rangle = \langle \frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\rangle$$

for some function $\psi(x,y)$. Function ψ is called *stream function* and its level curves coincide with the stream lines of the flow.

Now we also assume that the flow is *irrotational*, i.e., vector field $\mathbf{v}$ is a gradient vector field of a function $\varphi(x,y)$.

In this case,

$$\Delta\varphi = \Delta\psi = 0$$

are a pair of conjugate harmonic functions defining a holomorphic function

$$w(z) = \varphi(x,y) + \boldsymbol{i}\psi(x,y).$$

Example 6 (Flow around a disk)**.** Consider an infinitely long body immersed in a fluid. We can consider the problem to be two-dimensional, just as in the case of the coaxial cable. Imagine that before immersing the object, the flow was $\mathbf{v} = \langle 1, 0\rangle$. After the body is immersed, flow far away from the body is not disturbed, so the boundary conditions on φ and ψ are

$$\langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \rangle = \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle = \langle 1, 0\rangle.$$

The boundary condition at a solid static boundary is

$$\mathbf{v} \cdot \mathbf{n} = 0$$

which translates into a Neumann boundary condition for φ and Dirichlet boundary condition for ψ:

$$\mathbf{n} \cdot \nabla \varphi = 0, \quad \mathbf{t} \cdot \nabla \psi = 0$$

where $\mathbf{n}$ and $\mathbf{t}$ are normal and tangent vector fields at the boundary of the body. The latter equation implies that ψ is constant on the boundary of body.

Let us solve Dirichlet problem for ψ if we have a unit disk $\{|z| < 1\}$ placed in a flow. Potential for the initial flow $\mathbf{v} = \langle 1, 0\rangle$ is $\psi_0(x, y) = y$. Then function

$$u(z) := \psi(z) - y$$

is harmonic, equals zero at ∞ (since flows given by ψ_0 and ψ near ∞ are the same) and $-y$ on the boundary of the disk. Hence $u(z)$ solves an ordinary Dirichlet problem in the complement disk in $\widehat{\mathbb{C}}$ and has solution

$$u(z) = -\frac{y}{x^2 + y^2},$$

hence

$$\psi(z) = y\left(1 - \frac{1}{x^2 + y^2}\right)$$

and

$$\mathbf{v} = \left\langle 1 - \frac{x^2 - y^2}{(x^2 + y^2)^2}; -\frac{2xy}{(x^2 + y^2)^2} \right\rangle$$

Figure 1: Stream lines of the flow around the unit disk