Lecture 24

Elliptic functions

Let ω_1, ω_2 be two complex numbers such that ω_1 and ω_2 are linearly independent over $\mathbb{R}$.

Definition 1. Meromorphic function $f(z)$ is said to be *doubly periodic* if for any $z \in \mathbb{C}$

$$f(z+\omega_1) = f(z+\omega_2) = f(z).$$

Let $\tau := \omega_1/\omega_2$. Clearly $\tau \in \mathbb{C}$ is not purely real. Moreover, changing ω_1 to $-\omega_1$, if necessary, we can assume that $\mathfrak{Im}(\tau) > 0$. Considering function $F(z) := f(\omega_1 z)$ instead of f, we will get a doubly-periodic function withe periods 1 and τ. Hence, from now on we assume that periods of f are 1 and τ, so that

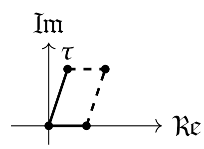
$$f(z+n+m\tau) = f(z), \quad \text{for all } n, m \in \mathbb{Z}.$$

We say that 1 and τ generate *lattice*

$$\Lambda := \{n + m\tau \mid n, m \in \mathbb{Z}\}.$$

and call two points $z, w \in \mathbb{C}$ equivalent modulo Λ if $z - w \in \Lambda$.

Let Π be the parallelogram generated by vectors 1 and τ:



This is *fundamental parallelogram*, i.e. any point $z \in \mathbb{C}$ is equivalent to a unique point in Π (you have to include only one vertex and two adjacent sides into Π for it to work). Therefore any doubly periodic function $f(z)$ is uniquely determined by its behavior on Π.

Remark 2. The same will be true for any translate of Π: a parallelogram of the form $\Pi + h$, $h \in \mathbb{C}$.

The following theorem sates that there is no interesting holomorphic doubly periodic functions.

Theorem 3. *An* ***entire*** *doubly periodic function is constant.*

Proof. Being continuous on a compact region $\overline{\Pi}$, function $|f(z)|$ is bounded. Therefore, by Liouville's theorem, $f(z)$ is constant. □

Therefore, to get interesting doubly-periodic functions, we have to allow for poles. A non-constant doubly-periodic function is called **elliptic function**. Such function must have finitely many poles in any given bounded set. In particular, there are finitely many poles in Π. It turns out, elliptic function $f(z)$ has to have at least two poles (or a double pole) in Π.

Theorem 4. *The total number of poles of an elliptic function $f(z)$ (counted with multiplicities) is $\geqslant 2$.*

Proof. If there are no poles on $\partial\Pi$, we can apply residue theorem:

$$\int_{\partial\Pi} f(z)dz = 2\pi \boldsymbol{i} \sum \operatorname{res}(f).$$

On the other hand, the integral on the left hand side is zero since $f(z)$ is doubly periodic (and integrals over the opposite sides of Π cancel out). Hence sum of the residues is 0, so $f(z)$ can not have only one simple pole in Π.

If there are poles on $\partial\Pi$, we can shift Π a bit by substituting it with $\Pi + h$ for some $h \in \mathbb{C}$. □

Definition 5. Number of poles (with multiplicities) of an elliptic function $f(z)$ is called its **order** (not to be confused with the order of a zero or a pole).

Theorem 6. *Elliptic function $f(z)$ of order m has exactly m zeros in Π.*

Corollary 7. *Elliptic function of order m takes any value $c \in \mathbb{C}$ exactly m times.*

Proof of the corollary. Consider zeros of an elliptic function $f(z) - c$. □

Proof of the theorem. Assume first that $f(z)$ does not have zeros or poles on $\partial\Pi$. Then by argument principle

$$\int_{\partial\Pi} \frac{f'(z)}{f(z)} dz = 2\pi \boldsymbol{i}(N_z - N_p),$$

where N_z and N_p are numbers of zeros and poles of $f(z)$. By periodicity, the integral must vanish, so since $N_p = m$, we must have $N_z = 0$. □

Up to this point, it remains an open question if elliptic functions exist.

Weierstrass $\wp$-function

Before discussing construction of $\wp$-function, let us recall the partial fraction expansion of $\pi\cot(\pi z)$. We knew that this function is $\mathbb{Z}$-periodic and that it has poles at all integers. It was natural to try to compare it to the function defined by the infinite sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n}$$

The problem is that this sum does not converge, and we had to tweak it by considering instead the sum

$$\frac{1}{z} + \sum_{n\in\mathbb{Z}, n\neq 0} \left(\frac{1}{z+n} - \frac{1}{n}\right).$$

On every compact domain, the n-th term of this sum can be bounded by C/n^2, hence the sum is absolutely and uniformly convergent on compact subsets $\mathbb{C} - \mathbb{Z}$.

The idea behind construction of $\wp(z)$ is to mimic the above to make sense of the divergent infinite double sum

$$\sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^2}.$$

We will follow a similar trick as for the infinite sum of $\pi\cot(\pi z)$, and consider an infinite double sum

$$\frac{1}{z^2} + \sum_{\omega\in\Lambda^*} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right), \tag{1}$$

where $\Lambda^* := \Lambda - \{(0,0)\}$. We note that

For the term in the brackets and z bounded, we have

$$\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} = \frac{-z^2 - 2z\omega}{(z+\omega)^2\omega^2} = O(1/\omega^3), \quad \omega \to \infty.$$

It is an easy exercise to verify at this point that the series in (1) is absolutely convergent. The function defined by this series is called **Weierstrass $\wp$ function**:

$$\wp(z) := \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)} \left(\frac{1}{(z+n+m\tau)^2} - \frac{1}{(n+m\tau)^2}\right).$$

Clearly $\wp(z)$ is even. Due to the uniform convergence, function $\wp(z)$ is meromorphic function with double poles at Λ. For the same reason, derivative of $\wp(z)$ can be computed by term-wise differentiation:

$$\wp'(z) = -2\sum_{\omega\in\Lambda}\frac{1}{(z+\omega)^3}. \tag{2}$$

Exercise 1. Show that function $\wp(z)$ is double periodic.

Hint: use the facts that $\wp(z)$ is even and $\wp'(z)$ is doubly periodic (due to the invariance of the sum (2) under shifts $z \mapsto z+\omega$

Properties of $\wp(z)$

Since $\wp(z)$ is even, and doubly periodic, it is easy to see that $\wp'(z)$ must vanish at the 'half-periods' $1/2, \tau/2$ and $(1+\tau)/2$. Since $\wp'(z)$ is of order 3, these must three simple zeros of $\wp'(z)$. In particular, if we set

$$\wp(1/2) = e_2, \quad \wp(\tau/2) = e_2, \quad \wp(\frac{1+\tau}{2}) = e_3,$$

we see that each of the function $\wp(z) - e_i$ has a double zero at the corresponding half-period. This implies that all the number e_1, e_2, e_3 are distinct, since $\wp(z) - c$ can have only two zeros with multiplicities for any given c.

The following theorem provides the key property of the $\wp$-function.

Theorem 8. *The function $(\wp')^2$ is the cubic polynomial in $\wp(z)$:*

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3).$$

Proof. The ratio of the left hand side and $(\wp - e_1)(\wp - e_2)(\wp - e_3)$ is a doubly periodic function with no zeros. Therefore, it must be a constant. Comparing the leading term of the Laurent series of $(\wp')^2$ and the right hand side we find, that the constant is 4. □

In a certain sense, $\wp$ and $\wp'$ are universal elliptic functions.

Theorem 9. *Every elliptic function f with periods 1 and τ is a rational function of $\wp$ and $\wp'$.*

Proof. Any elliptic function $f(z)$ is a sum of an odd and an even elliptic function:

$$f(z) = f_{\text{odd}}(z) + f_{\text{even}}(z) := \frac{f(z) - f(-z)}{2} + \frac{f(z) + f(-z)}{2}$$

and $f_{\text{odd}}/\wp'$ is also even. Therefore, it is enough to prove that any even elliptic function can be represented as a rational function of $\wp$. The idea is to use $\wp(z)$ to construct an elliptic function with prescribed zeros and poles.

Given any even elliptic $F(z)$, if $a \notin \Lambda$ is a zero of F, then $-a$ is also a zero, and a is equivalent to $-a$ modulo Λ if and only if a is a half period. In the latter case zero at a has even multiplicity.

Dividing F by $\wp(z) - \wp(a)$ we would kill these zeros at a and $-a$. Repeating this procedure we would kill all the zero in Π, introducing possibly zeros at points of Λ.

Similarly we can kill all poles of F outside Λ, by multiplying F with $\wp(z) - \wp(b)$. We end with a function which does not have any zeros or poles in $\mathbb{C} - \Lambda$. Such a function can not have a pole and a zero in Π, therefore it must be constant. □