Lecture 3

Examples of holomorphic functions

In the last lecture we saw that being a holomorphic function is a very restrictive condition. The aim of today's lecture is to construct a large supply of holomorphic functions.

Polynomials and Rational functions

If $f(z)$ and $g(z)$ are holomorphic function on their domains, then all the functions $f+g, f\cdot g, f/g, f\circ g$ are also holomorphic on their domains.

Since $f(z)=z$ is trivially holomorphic on $\mathbb{C}$, as an immediate consequence we conclude:

1. Power functions $f(z)=z^k, k\in\mathbb{N}$ are holomorphic on $\mathbb{C}$ with
$$f'(z)=kz^{k-1}$$
2. Polynomials $P(z)=\sum_{n=0}^d a_n z^n$ are holomorphic on $\mathbb{C}$ with
$$P'(z)=\sum_{n=0}^d n a_n z^{n-1}.$$
3. Rational functions $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are holomorphic everywhere on the domain $\mathcal{D} := \{z \mid Q(z)\neq 0\}$. By the Fundamental Theorem of Algebra $\mathcal{D}$ is the whole complex plane except for at most $\deg Q$ points.

By Fundamental Theorem of Algebra, any polynomial $P(z)$ can be written as $P(z)=a_n(z-w_1)\dots(z-w_n)$. Complex numbers $\{w_1,\dots,w_n\}$ are called *roots* or *zeros* of P.

Definition 1. If $R(z)=P(z)/Q(z)$ is an *irreducible* rational function (i.e. $P(z)$ and $Q(z)$ do not have common factors), then zeros of $P(z)$ are a *zeros* of $R(z)$, while zeros of $Q(z)$ are *poles* of $R(z)$. Poles β are characterized by the property
$$\lim_{z\to\beta} R(z)=\infty.$$

Order of a zero β is the number $k\in\mathbb{N}$ such that $R(z)/(z-\beta)^k$ has a finite non-zero limit as $z\to\beta$. Similarly *order* of a pole β is the number $k\in\mathbb{N}$ such that $R(z)(z-\beta)^k$ has a finite non-zero limit as $z\to\beta$.

It is convenient to extend the domain and range of a rational function to the Riemann sphere $\hat{\mathbb{C}}$:
$$R\colon \hat{\mathbb{C}}\to\hat{\mathbb{C}}.$$

Concretely, to define $R(\infty)$ we consider $R_1(z) := R(1/z)$ and set $R(\infty) := R_1(0)$.

Theorem 2 (Partial Fraction Expansion). *Given a rational function $R(z)$ with poles $\beta_1,\dots,\beta_k\in\mathbb{C}$, there exist polynomials $G(z)$ and $G_j(z), j=1,\dots k$ such that*
$$R(z)=G(z)+\sum_{j=1}^k G_j\left(\frac{1}{z-\beta_j}\right).$$

Proof. Let β be one of the poles of $R(z)$ of order k, i.e.,
$$R(z)=\frac{\widetilde{R}(z)}{(z-\beta)^k},$$
where $\widetilde{R}(\beta)$ is a non-zero complex number. Then the rational function
$$R(z)-\frac{\widetilde{R}(\beta)}{(z-\beta)^k}=\frac{\widetilde{R}(z)-\widetilde{R}(\beta)}{(z-\beta)^k}$$

has pole at β of order $< k$.

Iterating this procedure we will eventually eliminate all poles $\beta_1, \ldots, \beta$ by subtracting expression of the form

$$\sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right).$$

We are left with a rational function

$$R(z) - \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right)$$

with no poles. A rational function with no poles in $\mathbb{C}$ is a polynomial $G(z)$, therefore

$$R(z) = G(z) + \sum_{j=1}^{k} G_j\left(\frac{1}{z-\beta_j}\right).$$

as required. □

The set of rational functions $\{P(z)/Q(z)\}$ is closed under all basic operations: addition, multiplication, division, composition. Therefore we need some new techniques to construct another examples of holomorphic functions.

Power series

In this section we review the basic theory of power series which is the most important, and essentially, the only source of holomorphic functions.

Definition 3. A (complex) power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$. A power series *converges* at z_0 if there exists a finite limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n {z_0}^n.$$

A power series *absolutely converges* at z_0 if there exists a finite limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} |a_n||z_0|^n.$$

Remark 4. By *Cauchy's convergence test*, if a power series absolutely converges at z_0, then it converges at z_0.

Remark 5. If a power series absolutely converges at z_0, then it will also absolutely converge at any z' with $|z'| \leqslant |z_0|$.

Example 6. The prime example of a power series is the power series representing the exponential function:

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We will prove that the series on the right hand side yields a well-defined holomorphic function $\mathbb{C} \to \mathbb{C} - \{0\}$.

Theorem 7. *Given a power series*

$$\sum_{n=0}^{\infty} a_n z^n,$$

Let $0 \leqslant R \leqslant +\infty$ *such that*

- *If $|z| < R$ the series converges absolutely.*
- *If $|z| > R$ the series diverges.*

Moreover, R is given by Hadamard's formula:

$$1/R = \limsup |a_n|^{1/n}.$$

Number R is called the radius of convergence of $\sum a_n z^n$.

Proof. Let $1/R$ be the number defined by Hadamard's formula. Given $|z| < R$, we can choose $\epsilon > 0$ such that the number

$$r := (1/R + \epsilon)|z| < 1.$$

By definition of $1/R$ we have that for all n large enough

$$|a_n|^{1/n} < (1/R + \epsilon) \Leftrightarrow |a_n| < (1/R + \epsilon)^n.$$

Therefore, for n large enough we have

$$|a_n||z^n| < ((1/R + \epsilon)|z|)^n = r^n.$$

Hence the 'tail' of $\sum |a_n||z|^n$ is dominated by a convergent geometric series.

The part $|z| > R$ is left as an exercise. □

Example 8. For the series defining e^z we have

$$1/R = \limsup |1/n!|^{1/n} = 0,$$

since $n! \geqslant n^{n/2}$. Hence $R = +\infty$ and e^z is defined by an absolutely convergent power series.

Any power series is holomorphic in its disc of convergence $B_R(0)$.

Theorem 9. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in the open disc $B_R(0)$, where R is the radius of convergence. Moreover, $f'(z)$ is given by the power series with the same radius of convergence obtained from $\sum_{n=0}^{\infty} a_n z^n$ by the term-wise differentiation:*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Proof. First we note that since $\lim_{n\to\infty} n^{1/n} = 0$,

$$\limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n},$$

so that $\sum a_n z^n$ and $\sum n a_n z^n$ have the same radius of convergence, hence so does $\sum n a_n z^{n-1}$.

Denote

$$g(z) := \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Not take z_0 with $|z_0| < r < R$. Our aim is to prove that the difference

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right|$$

can be made arbitrary small by choosing h small enough.

Let us break the series defining $f(z)$ into two parts:

$$f(z) = S_N(z) + E_N(z) = \left(\sum_{n=0}^{N} a_n z^n \right) + \left(\sum_{n=N+1}^{\infty} a_n z^n \right)$$

with N to be determined. Then for h such that $|z_0 + h| < r$ we can rewrite

$$\frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0 + h) - S(z_0)}{h} - S_N'(z_0) \right) + (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0 + h) - E_N(z_0)}{h} \right).$$

We want to bound all three terms on the right hand side.

1. Since $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) \leqslant n|a-b|\max(|a|,|b|)^{n-1}$, we have for the third summand

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leqslant \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| < \sum_{n=N+1}^{\infty} n|a_n|r^{n-1}.$$

The final expression is the tail a convergent series, since $g(z)$ absolutely converges in $\{z \mid |z| < R\}$. Hence given $\epsilon > 0$ we can find N_1 large enough so that for $N > N_1$

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \epsilon.$$

2. Next, since $\lim_{N\to\infty} S'_N(z_0) = g(z_0)$, we can find N_2 so that for $N > N_2$

$$|S'_N(z_0) - g(z_0)| < \epsilon.$$

Fix $N > \max(N_1, N_2)$

3. Finally, since $S'_N(z_0)$ is the complex derivative of a polynomial $S_N(z)$ at $z = z_0$, we can find $\delta > 0$ such that for $|h| < \delta$ we have

$$\left|\frac{S_N(z_0+h) - S(z_0)}{h} - S'_N(z_0)\right| < \epsilon.$$

Collecting three inequalities together we find:

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)\right| < 3\epsilon$$

Since ϵ is arbitrary, we conclude that $g(z_0)$ is the derivative of $f_0(z)$ at $z = z_0$. □

Corollary 10. *A power series $f(z) = \sum a_n z^n$ is infinitely complex differentiable in its disk of convergence, and all its derivatives could be computed by the term-wise differentiation. In particular*

$$a_p = \frac{f^{(p)}(0)}{p!}, p \in \mathbb{N}.$$

Example 11. Applying the above theorem to the series defining e^z, we conclude that e^z is holomorphic with $(e^z)' = e^z$.

Exercise 1. Prove that for $z, w \in \mathbb{C}$ we have

$$e^z \cdot e^w = e^{z+w}.$$

Hint: multiply the series defining e^z and e^w. Using the absolute convergence rearrange the terms in the resulting double-sum.

Since $e^0 = 1$, the above exercise implies that $e^z \cdot e^{-z} = 1$, so $e^z \neq 0$.

Exercise 2. Define

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}.$$

Then both functions are holomorphic on $\mathbb{C}$ and their derivatives are given by

$$\cos'(z) = -\sin(z), \quad \sin'(z) = \cos(z).$$

Complex logarithm

Multivalued logarithm

Given $w = x + iy \in \mathbb{C}$ let us now try to solve equation $w = e^z$ for z. If $w = 0$, then the equation has no solution, so let from now on assume $w \neq 0$.

- $|w|^2 = w \cdot \overline{w} = e^z \cdot e^{\overline{z}} = e^{2\mathfrak{Re} z}$. Hence we find $\mathfrak{Re}(z) = \log|w|$, where $\log = \log_e \colon \mathbb{R}_{>0} \to \mathbb{R}$ is the usual logarithmic function.

- Since $e^{\Re\mathfrak{e} z} = |w|$, we have $w/|w| = e^{\boldsymbol{i}\,\Im\mathfrak{m} z}$. This equation has infinitely many solutions

  $$\Im\mathfrak{m} z = \varphi + 2\pi k, k \in Z,$$

  where $\varphi := \mathrm{Arg} w \in (-\pi, \pi]$ is the *principle branch* of the argument of w.

The above observation allows us to define a '*multivalued function*' (this is not a function in the usual sense)

$$\log w := \log|w| + \boldsymbol{i}\arg w,$$

where $\arg w$ is the multivalued argument of w. Any two values of $\log w$ differ by a multiple of $\boldsymbol{i}2\pi$.

Principle branch

Often it is inconvenient to work with multivalued functions. To this end we will fix the *principle branch of logarithm* by setting

$$\mathrm{Log}\, w := \log|w| + \boldsymbol{i}\mathrm{Arg} w.$$

This way logarithm becomes a single valued function $\mathbb{C} - \{0\} \to \mathbb{C}$. The main drawback of this definition is that Log is discontinuous along the negative ray $\{z = x + \boldsymbol{i} \cdot 0 \mid x < 0\}$: once we move from $x + \boldsymbol{i}\epsilon$ to $x - \boldsymbol{i}\epsilon$, the value of $\Im\mathfrak{m}\mathrm{Log}$ jumps by 2π. To 'fix' this issue, sometimes we will reduce the domain of Log and consider it as a function

$$\mathrm{Log}\colon \mathbb{C} - \{z = x + \boldsymbol{i} \cdot 0 \mid x < 0\} \to \mathbb{C}.$$

Using Log we can define fractional and even any complex power of a complex number $z \in \mathbb{C} - \{z = x + \boldsymbol{i} \cdot 0 \mid x < 0\}$:

$$z^w := e^{w\mathrm{Log} z}.$$

Of course, instead of making a *cut* along the ray $\{z = x + \boldsymbol{i} \cdot 0 \mid x < 0\}$ we could make a cut along any other ray $\{z = e^{\boldsymbol{i}\varphi} x \mid x > 0\}$.

Remark 12. Choosing a branch of the logarithmic function we inevitably loose the key property

$$\log z + \log w = \log(zw).$$

Instead, this identity holds only up to summands of the form $2\pi\boldsymbol{i}k, k \in N$:

$$\mathrm{Log} z + \mathrm{Log} w = \mathrm{Log}(zw) + 2\pi\boldsymbol{i}k.$$

More generally, one can define a single-valued logarithmic function in any open *simply-connected* region $\Omega \subset \mathbb{C}$ provided $0 \notin \Omega$. This will be done in future lectures.