Lecture 4

Power series continued

Any power series is holomorphic in its disc of convergence $B_R(0)$.

Theorem 1. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in the open disc $B_R(0)$, where R is the radius of convergence. Moreover, $f'(z)$ is given by the power series with the same radius of convergence obtained from $\sum_{n=0}^{\infty} a_n z^n$ by the term-wise differentiation:*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}.$$

Proof. First we note that since $\lim_{n\to\infty} n^{1/n} = 0$,

$$\limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n},$$

so that $\sum a_n z^n$ and $\sum n a_n z^n$ have the same radius of convergence, hence so does $\sum n a_n z^{n-1}$.

Denote

$$g(z) := \sum_{n=0}^{\infty} n a_n z^{n-1}$$

Not take z_0 with $|z_0| < r < R$. Our aim is to prove that the difference

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)\right|$$

can be made arbitrary small by choosing h small enough.

Let us break the series defining $f(z)$ into two parts:

$$f(z) = S_N(z) + E_N(z) = \left(\sum_{n=0}^{N} a_n z^n\right) + \left(\sum_{n=N+1}^{\infty} a_n z^n\right)$$

with N to be determined. Then for h such that $|z_0 + h| < r$ we can rewrite

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right).$$

We want to bound all three terms on the right hand side.

1. Since $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}) \leqslant n|a-b|\max(|a|,|b|)^{n-1}$, we have for the third summand

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leqslant \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| < \sum_{n=N+1}^{\infty} n|a_n| r^{n-1}.$$

The final expression is the tail a convergent series, since $g(z)$ absolutely converges in $\{z \mid |z| < R\}$. Hence given $\epsilon > 0$ we can find N_1 large enough so that for $N > N_1$

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \epsilon.$$

2. Next, since $\lim_{N\to\infty} S_N'(z_0) = g(z_0)$, we can find N_2 so that for $N > N_2$

$$|S_N'(z_0) - g(z_0)| < \epsilon.$$

Fix $N > \max(N_1, N_2)$

3. Finally, since $S_N'(z_0)$ is the complex derivative of a polynomial $S_N(z)$ at $z = z_0$, we can find $\delta > 0$ such that for $|h| < \delta$ we have

$$\left|\frac{S_N(z_0+h) - S(z_0)}{h} - S_N'(z_0)\right| < \epsilon.$$

Collecting three inequalities together we find:

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g(z_0)\right| < 3\epsilon$$

Since ϵ is arbitrary, we conclude that $g(z_0)$ is the derivative of $f_0(z)$ at $z = z_0$. ☐

Corollary 2. *A power series $f(z) = \sum a_n z^n$ is infinitely complex differentiable in its disk of convergence, and all its derivatives could be computed by the term-wise differentiation. In particular*

$$a_p = \frac{f^{(p)}(0)}{p!}, p \in \mathbb{N}.$$

Example 3. Applying the above theorem to the series defining e^z, we conclude that e^z is holomorphic with $(e^z)' = e^z$.

Exercise 1. Prove that for $z, w \in \mathbb{C}$ we have

$$e^z \cdot e^w = e^{z+w}.$$

Hint: multiply the series defining e^z and e^w. Using the absolute convergence rearrange the terms in the resulting double-sum.

Since $e^0 = 1$, the above exercise implies that $e^z \cdot e^{-z} = 1$, so $e^z \neq 0$.

Exercise 2. Define

$$\cos(z) := \frac{e^{\boldsymbol{i}z} + e^{-\boldsymbol{i}z}}{2}, \quad \sin(z) := \frac{e^{\boldsymbol{i}z} - e^{-\boldsymbol{i}z}}{2\boldsymbol{i}}.$$

Then both functions are holomorphic on $\mathbb{C}$ and their derivatives are given by

$$\cos'(z) = -\sin(z), \quad \sin'(z) = \cos(z).$$

Complex logarithm

Multivalued logarithm

Given $w = x + \boldsymbol{i}y \in \mathbb{C}$ let us solve equation $w = e^z$ for z. If $w = 0$, then the equation has no solution, so from now on we assume $w \neq 0$.

- $|w|^2 = w \cdot \overline{w} = e^z \cdot e^{\overline{z}} = e^{2\mathfrak{Re} z}$. Hence we find $\mathfrak{Re}(z) = \log|w|$, where $\log = \log_e \colon \mathbb{R}_{>0} \to \mathbb{R}$ is the usual logarithmic function.
- Since $e^{\mathfrak{Re} z} = |w|$, we have $w/|w| = e^{\boldsymbol{i}\,\mathrm{Im} z}$. This equation has infinitely many solutions

  $$\mathfrak{Im} z = \varphi + 2\pi k, k \in Z,$$

  where $\varphi := \mathrm{Arg} w \in (-\pi, \pi]$ is the *principle branch* of the argument of w.

The above observation allows us to define a '*multivalued function*' (this is not a function in the usual sense)

$$\log w := \log|w| + \boldsymbol{i}\arg w,$$

where $\arg w$ is the multivalued argument of w. Any two values of $\log w$ differ by a multiple of $\boldsymbol{i}2\pi$.

Principle branch

Often it is inconvenient to work with multivalued functions. To this end we will fix the *principle branch of logarithm* by setting

$$\mathrm{Log}\, w := \log|w| + \boldsymbol{i}\mathrm{Arg} w.$$

This way logarithm becomes a single valued function $\mathbb{C} - \{0\} \to \mathbb{C}$. The main drawback of this definition is that Log is discontinuous along the negative ray $\{z = x + \boldsymbol{i} \cdot 0 \mid x < 0\}$: once we move from $x + \boldsymbol{i}\epsilon$ to $x - \boldsymbol{i}\epsilon$, the value of $\mathfrak{Im}\mathrm{Log}$ jumps by 2π. To 'fix' this issue, sometimes we will reduce the domain of Log and consider it as a function

$$\mathrm{Log} \colon \mathbb{C} - \{z = x + \boldsymbol{i} \cdot 0 \mid x \leqslant 0\} \to \mathbb{C}.$$

Using Log we can define fractional and even any complex power of a complex number $z \in \mathbb{C} - \{z = x + \boldsymbol{i} \cdot 0 \mid x \leqslant 0\}$:

$$z^w := e^{w \mathrm{Log} z}.$$

Of course, instead of making a *cut* along the ray $\{z = x + \boldsymbol{i} \cdot 0 \mid x \leqslant 0\}$ we could make a cut along any other ray $\{z = e^{\boldsymbol{i}\varphi} x \mid x \geqslant 0\}$.

Remark 4. Choosing a branch of the logarithmic function we inevitably loose the key property

$$\log z + \log w = \log(zw).$$

Instead, this identity holds only up to summands of the form $2\pi \boldsymbol{i} k, k \in N$:

$$\operatorname{Log} z + \operatorname{Log} w = \operatorname{Log}(zw) + 2\pi \boldsymbol{i} k.$$

More generally, one can define a single-valued logarithmic function in any open *simply-connected* region $\Omega \subset \mathbb{C}$ provided $0 \notin \Omega$. This will be done in future lectures.

Integration along curves

Definition 5. A *parametrized curve* is a function $z(t)$ which maps a closed interval $[a, b] \in \mathbb{R}$ to $\mathbb{C}$. We will always impose regularity conditions on $z(t)$. We say that the curve is *smooth* if $z'(t)$ exists and continuous and $z'(t) \neq 0$ for $t \in [a, b]$.

All curves in this course will be continuous and piecewise smooth.

Two parametrizations $z\colon [a, b] \to \mathbb{C}$ and $\widetilde{z}\colon [\alpha, \beta] \to \mathbb{C}$ are *equivalent* if there exist a continuously differentiable bijection (reparametrization) $s \to t(s)$ with $t'(s) > 0$ such that:

$$\widetilde{z}(s) = z(t(s)).$$

The equivalence class of a parametrized curve is a *plane curve* $\gamma \subset \mathbb{C}$ with a fixed *orientation*.

We define the integral of a complex-valued function $f(z)$ along a curve γ parametrized by $z\colon [a, b] \to \mathbb{C}$ as:

$$\int_\gamma f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

Provided $f(z)$ is continuous, the integral on the right-hand side is well-defined.

Proposition 6. *The above integral does not depend on parametrization.*

Proof. If $\widetilde{z}(s) := z(t(s))$ is another smooth parametrization of the same oriented plane curve then by the change of variables

$$\int_a^b f(z(t))z'(t)dt = \int_\alpha^\beta f(z(t(s)))z'(t(s))t'(s)ds = \int_\alpha^\beta f(\widetilde{z}(s))\widetilde{z}'(s)ds.$$

□

Next is an extremely important example.

Example 7. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{\boldsymbol{i}t}$ be the unit circle traversed counterclockwise. Then

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{\boldsymbol{i} e^{\boldsymbol{i}t} dt}{e^{\boldsymbol{i}t}} = \int_0^{2\pi} \boldsymbol{i} dt = 2\pi \boldsymbol{i}.$$

Exercise 3. For $\gamma\colon [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{\boldsymbol{i}t}$ the unit circle traversed counterclockwise compute

$$\int_\gamma z^k dz, \qquad k \in \mathbb{Z}.$$

Elementary properties of integration

Complex integration along curves satisfies many familiar properties.

- (Orientation) Given a curve γ, let $(-\gamma)$ be the curve traversed in the opposite direction. Then

$$\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz$$

- (Linearity in functions) Given complex valued functions $f(z), g(z)$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\int_\gamma (\alpha f(z) + \beta g(z)) dz = \alpha \int_\gamma f(z) dz + \beta \int_\gamma g(z) dz.$$

- (Linearity in curves) If $\gamma = \gamma_1 \cup \gamma_2$ is a subdivision of a curve γ, then

$$\int_\gamma f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz.$$

Definition 8. Let $f(z)$ be a complex-valued function. A *primitive* of $f(z)$ is a holomorphic function $F(z)$ such that $F'(z) = f(z)$.

Theorem 9 (Fundamental Theorem of Calculus)**.** *Let $F(z)$ be a primitive of a continuous $f(z)$ in an open region $U \subset \mathbb{C}$. Then for any oriented curve $\gamma \subset U$ with endpoints z_0 and z_1*

$$\int_\gamma f(z) dz = F(z_1) - F(z_0).$$

Proof. Let $z(t) \colon [0,1] \to \mathbb{C}$ be a parametrization of γ. Then

$$\int_\gamma f(z) dz = \int_0^1 F'(z(t)) z'(t) dt = \int_0^1 \frac{dF(z(t))}{dt} dt = F(z_1) - F(z_0),$$

where in the last step we used the usual Fundamental Theorem of Calculus applied to real and imaginary parts of $t \mapsto F(z(t))$. □

Corollary 10. *Function $f(z) = 1/z$ does not have a primitive in $\mathbb{C} - \{0\}$.*

Next theorem provides the converse of the FTC.

Theorem 11. *Assume that a continuous complex-valued function $f(z) \colon U \to \mathbb{C}$ satisfies*

$$\int_\gamma f(z) dz = \int_{\gamma'} f(z) dz$$

where $\gamma, \gamma' \subset U$ are two curves with the same starting and end points. Then $f(z)$ admits a primitive $F(z)$.

Proof. Let $U_0 \subset U$ be a connected component of U, and let $z_0 \in U_0$ be a base point. For any $z \in U_0$ consider a curve γ from z_0 to z end define

$$F(z) := \int_\gamma f(z) dz.$$

By the Theorem's assumption, the above definition is independent of curve γ.

We claim that $F(z)$ defined by the above formula is holomorphic with $F'(z) = f(z)$. Indeed, by the definition of $F(z)$ we have:

$$\frac{F(z+h) - F(z)}{h} = \frac{1}{h} \int_{\gamma_h} f(z) dz,$$

where γ_h is any curve connecting z to $z+h$. By choosing γ_h to be the straight segment connecting z to $z+h$ and using the continuity of $f(z)$ it is easy to see that the limit of this expression as $h \in C$ goes to 0 is $f(z)$. Hence $F(z)$ is indeed a primitive of $f(z)$ in U_0.

If $\tilde{F}(z)$ is another primitive of $f(z)$ in U_0 then $F(z)$ and $\tilde{F}(z)$ differ by a constant since $(F(z) - \tilde{F}(z)$ has vanishing complex derivative, in particular, its real and imaginary parts are constants.

Applying the above argument to every connected component of U we prove the theorem. □