Lecture 5

Cauchy's theorem

Today we will prove the most important result of complex analysis, which the key to many other theorems of the course, including analyticity of holomorphic functions, Liouville's theorem, and calculus of residues.

Last time we saw the a continuous complex-valued function $f(z)$ has a primitive in some open region U if and only if the integral

$$\int_\gamma f(z)dz$$

is independent of the choice of path γ between any points $z_0, z_1 \in U$. Today we will show that, under certain assumptions on U the latter condition holds for any holomorphic function $f(z)$. In particular, such $f(z)$ will have a primitive in U.

We start with the Goursat's theorem, which is essentially the Cauchy's theorem for triangles. From it in effect we will deduce all other versions of Cauchy's theorem.

Theorem 1 (Goursat's theorem)**.** *If $f(z)$ is holomorphic in U and $T \subset U$ is a triangle contained in U with its interior, then*

$$\int_T f(z)dz = 0.$$

Proof. Let $T = T^{(0)}$ be the initial triangle and divide it into 4 smaller triangles $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}, T_4^{(1)}$ by its mid-lines. With appropriate orientations of the boundaries of these triangles, we have

$$\int_{T^{(0)}} f(z)dz = \sum_{j=1}^{4} \int_{T_j^{(1)}} f(z)dz.$$

Repeating this procedure with smaller triangles $N-1$ times, we will split the large triangle into 4^N small ones $\{T_j^{(N)}\}_{j=1}^{4^N}$ with the identity

$$\int_{T^{(0)}} f(z)dz = \sum_{j=1}^{4^N} \int_{T_j^{(N)}} f(z)dz.$$

Let $T^{(N)}$ be the triangle among $\{T_j^{(N)}\}_{j=1}^{4^N}$ for which the integral has the maximal absolute value. In this case we obviously have

$$\left|\int_{T^{(0)}} f(z)dz\right| \leqslant 4^N \left|\int_{T^{(N)}} f(z)dz\right|$$

We proceed indefinitely and obtain a sequence of nested triangles. By a modification of the nested intervals Theorem we find a unique point z_0 belonging to the interior of all triangles $T^{(N)}$.

Since $f(z)$ is holomorphic at z_0, we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

where $\psi(z) \to 0$ as $z \to z_0$.

Fix arbitrary $\epsilon > 0$ choose N large enough so that in the interior of $T^{(N)}$

$$|\psi(z)| < \epsilon.$$

Since both the constant term $f(z_0)$ and the linear term $f'(z_0)(z-z_0)$ have primitives, their integrals vanish, and we have

$$\int_{T^{(N)}} f(z)dz = \int_{T^{(N)}} \psi(z)(z - z_0)dz.$$

On the other hand, the integral can be bounded as

$$\left|\int_{T^{(N)}} \psi(z)(z - z_0)dz\right| \leqslant (\text{Length of } T^{(N)}) \cdot (\text{maximum of } |\psi(z)|) \cdot (\text{maximum of } |z - z_0|) \leqslant p_N \cdot \epsilon \cdot d_N,$$

where p_N and d_N are the perimeter and diameter of $T^{(N)}$. It remains to note that $p_N = 2^{-N}p_0$ and $d_N = 2^{-N}d_0$, therefore

$$\left|\int_{T^{(0)}} f(z)dz\right| \leqslant 4^N\left|\int_{T^{(N)}} f(z)dz\right| \leqslant \epsilon p_0 d_0.$$

Since $\epsilon > 0$ is arbitrary, the integral is in fact zero. □

Remark 2. The classical proof of Cauchy's theorem assumed continuity of $f'(z)$ and is an easy consequence of Cauchy-Riemann equations and Green's theorem, which states that for continuously differentiable $f(u(x,y)$ and $v(x,y))$ in a *nice* region D with boundary C we have

$$\int_C udx + vdy = \int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy.$$

The crucial advantage of Goursat's theorem is that we need to assume only that $f'(z)$ exists.

Corollary 3. *The conclusion of Goursat's theorem is still true if we replace triangle T with a rectangular region R, or, more generally, with any polygonal region P enclosed by a closed polygonal chain.*

Proof. Triangulate the region and orient appropriately all triangles so that $\int_P = \sum\int_{T_i}$. □

One of the important consequences of Cauchy's theorem is the existence of primitives.

Theorem 4. *A holomorphic function in an open disk has a primitive in that disk.*

Proof. With Goursat's theorem as hand, we can essentially reproduce the proof from the last lecture. Let γ_z be a straight line segment connecting 0 to z. Define a function in our disk D by

$$F(z) := \int_{\gamma_z} f(w)dw.$$

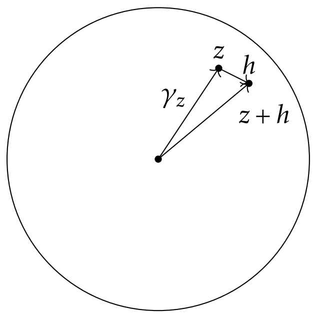

Figure 1

Then, since the integral of $f(z)$ over any triangle contained inside D is 0, we have

$$F(z+h) - F(z) = \int_\eta f(w)dw,$$

where η is the straight segment connecting z to $z+h$. Since f is continuous, we can write $f(w) = f(z) + \psi(w)$, where $\psi(w) \to 0$ as $\psi \to z$, hence

$$F(z+h) - F(z) = \int_\eta f(w)dw = \int_\eta f(z)dw + \int_\eta \psi(w)dw = f(z)h + o(h).$$

Letting $h \to 0$ we obtain $F'(z) = f(z)$. □

Theorem 5 (Cauchy's theorem in the disk)**.** *If f is holomorphic in a disk, then*

$$\int_\gamma f(z)dz = 0$$

for any closed curve γ in the disk.

Proof. From the previous theorem we know that $f(z)$ has a primitive. Now the conclusion follows from Theorem 9 (Fundamental Theorem of Calculus) from Lecture 4. □

We will also need the following improved version of the Cauchy theorem.

Theorem 6 (Improved Cauchy's theorem in the disk)**.** *Let f be a holomorphic function in a region D' obtained from an open disk D by removing a finite set of points $\{\zeta_i\}$. If $f(z)$ satisfies the conditions*

$$\lim_{z\to\zeta_j}(z-\zeta_j)f(z)=0, \qquad j=1,\dots n,$$

then

$$\int_\gamma f(z)dz=0$$

for any curve $\gamma\in D'$.

Proof. The proof of Theorem 4 must be modified as γ_z may pass through one of the points ζ_i. If γ_z passes through a point ζ_i we replace it with a *broken* segment γ_z' avoiding ζ_i. The key now is to prove that the value of the integral $\int_{\gamma_z'} f(z)dz$ is independent of the choice of γ_z'.

Remark 7. As shows the example of the integral $\int_{|z|=1}\frac{dz}{z}$ this is a non-trivial statement.

Enclose point ζ_i by a small square T centered at ζ_i. Assume that γ_z'' and γ_z' are two *broken paths* such that the region enclosed by γ_z',γ_z'' and T does not contain any singular points ζ_j

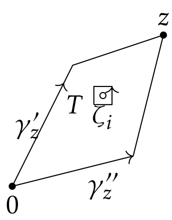

Figure 2

By triangulating this region and using the usual Gursat's theorem we conclude:

$$\int_{\gamma_z'} f(z)dz-\int_{\gamma_z''} f(z)dz=\int_T f(z)dz.$$

Using the property of $f(z)$ we can choose T small enough such that

$$|f(z)(z-\zeta_i)|<\epsilon$$

in T. Then

$$\left|\int_T f(z)dz\right|=\left|\int_T f(z)(z-\zeta)\frac{dz}{z-\zeta}\right|\leqslant\epsilon\int_T\frac{|dz|}{|z-\zeta|}=\epsilon\cdot C$$

where C is a constant independent of the choice of T.

Since ϵ is arbitrary, we conclude that $\int_T=0$, hence $\int_{\gamma_z'}=\int_{\gamma_z''}$ and we can construct a primitive of $f(z)$ in D'. Finally, since $f(z)$ has a primitive, integrals over closed contours in D' must vanish. □