Lecture 6

Index of a point with respect to a curve in $\mathbb{C}$

To formulate further results in complex analysis we will need to review some *topology notions* for plane curves.

Let γ be a closed, possibly self-intersecting piecewise smooth curve in $\mathbb{C}$. Given any point z_0 not on γ we can define an integral

$$\int_\gamma \frac{dz}{z-z_0}.$$

Proposition 1. *The above integral equals $2\pi \boldsymbol{i} n$, where $n = n(\gamma, z_0)$ is an integer.*[1]

Proof. One might try to prove this formula by considering the primitive of $1/(z-z_0)$, namely $\log(z-z_0)$. In this case,

$$\int_\gamma \frac{dz}{z-z_0} = \int_\gamma (d\log|z-z_0| + \boldsymbol{i}\arg(z-z_0)).$$

When the point on the curves travels along γ, function $\log|z-z_0| = \mathfrak{Re}\log(z-z_0)$ returns to its initial value, while $\arg(z-z_0) = \mathfrak{Im}\log(z-z_0)$ accumulates several multiples of 2π. Although this argument gives some intuition regarding the proposition, it is not rigorous: $\log(z-z_0)$ being a multi-valued function is not a well-defined primitive of $1/(z-z_0)$.

To give a rigorous proof, define

$$h(t) := \int_a^t \frac{z'(t)}{z(t)-z_0} dt, \quad t \in [a,b]$$

where $z(t), t \in [a,b]$ is a parametrization of γ. Clearly

$$h'(t) = \frac{z'(t)}{z(t)-z_0} \iff \frac{d}{dt}\left(e^{-h(t)}(z(t)-z_0)\right) = 0.$$

The latter implies that $e^{-h(t)}(z(t)-z_0)$ is constant in t, in particular

$$e^{h(t)} = \frac{z(t)-z_0}{z(a)-z_0}.$$

Since the curve is closed, we have $z(a) = z(b)$, in particular $e^{h(b)} = 1$. This implies that there exists integer n such that

$$\int_\gamma \frac{dz}{z-z_0} = h(b) = 2\pi \boldsymbol{i} n.$$

□

Definition 2. The number $n = n(\gamma, z_0)$ which we defined in the above proposition is called *the index of point z_0 with respect to curve γ*

$$n(\gamma, z_0) := \int_\gamma \frac{dz}{z-z_0}$$

The number $n(\gamma, z_0)$ measures how many times the curve γ winds around z_0 in positive (counterclockwise) direction. For this reason, often $n(\gamma, z_0)$ is also often called *winding number*.

Example 3. Let γ be the unit circle traveled in the positive direction. We know that

$$\int_\gamma \frac{dz}{z} = 2\pi \boldsymbol{i}.$$

Hence $n(\gamma, 0) = 1$.

Let us record several properties of the winding number

Proposition 4. *The winding numbers satisfy the following properties:*

[1] This is a manifestation of the fact that the first cohomology group of $\mathbb{C} - \{z_0\}$ is $\mathbb{Z}$: $H^1(\mathbb{C} - \{z_0\}; \mathbb{Z}) = \mathbb{Z}$.

1. *$n(-\gamma, z_0) = -n(\gamma, z_0)$*
2. *If $\gamma \subset B_R(w)$, and $z_0 \notin B_R(w)$, then $n(\gamma, z_0) = 0$.*
3. *As a function of z_0, the index $n(\gamma, z_0)$ is constant in every region cut out by γ and is zero in the unbounded region.*

Proof. Part (1.) is trivial, since reversing the orientation of a curve γ, we also reverse the sign of the integral defining $n(\gamma, z_0)$.

To prove part (2.) we note that by our assumption $f(z) = 1/(z - z_0)$ is holomorphic in $B_R(w)$, therefore by Cauchy's theorem integral of $f(z)$ along any closed loop in $B_R(w)$ is zero.

To prove the last part (3.) on could argue in two ways. One proof would just to note that $n(\gamma, z_0)$ is a continuous function of z_0, and since the index takes only integer values it has to be locally constant.

The other, a bit more formal proof might go like this. Let $a, b \in \mathbb{C}$ be two points in one connected component cut out be γ. Then we can connect a to b by a chain of segments which does not intersect γ. By induction we can assume that this chain actually consists of one segment, and we want to prove that

$$n(\gamma, a) = n(\gamma, b) \Longleftrightarrow \int_\gamma \Big(\frac{1}{z-a} - \frac{1}{z-b}\Big) dz = 0.$$

It follows from the homework exercise that the integrand $g(z) = \frac{1}{z-a} - \frac{1}{z-b}$ has a primitive in the complement of the segment joining a and b: $\mathbb{C} - \{ta + (1-t)b \,|\, t \in [0,1]\}$. Therefore, since $g(z)$ has a primitive in a neighbourhood of γ, by Fundamental Theorem of Calculus we conclude that

$$\int_\gamma g(z) dz = 0.$$

The last part of (3.) follows from (2.). Indeed, we can find a large disk D such that $\gamma \subset D$. Then for any $w \notin D$ we have $n(\gamma, w) = 0$. Since $n(\gamma, w)$ is locally constant, the same is true for the entire unbounded region cut out by γ. □

Remark 5. Let γ_s be a *smooth family* of closed curves. That is there exists a joint smooth parametrization:

$$z(s,t) \colon [0,1] \times [a,b] \to \mathbb{C}$$

such that for any fixed $s_0 \in [0,1]$, the curve γ_{s_0} is described by $z(s_0, t)$. Assume that point z_0 is not in the range of $z(s,t)$. Then

$$n(\gamma_0, z_0) = n(\gamma_1, z_0).$$

Indeed $n(\gamma_s, z_0)$ being a continuous function of s has to be locally constant. Therefore $n(\gamma_0, z_0) = n(\gamma_1, z_0)$.

This shows that the winding number $n(\gamma, z_0)$ is invariant under *smooth homotopy* of the curve γ.

Cauchy's integral formula

Last time we have proved the following result.

Theorem 6. *[Improved Cauchy's theorem in the disk] Let f be a holomorphic function in a region D' obtained from an open disk D by removing a finite set of points $\{\zeta_i\}$. If $f(z)$ satisfies the conditions*

$$\lim_{z \to \zeta_j} (z - \zeta_j) f(z) = 0, \qquad j = 1, \ldots n,$$

then

$$\int_\gamma f(z) dz = 0$$

for any curve $\gamma \in D'$.

Now we use this theorem together with the notion of a winding number to get Cauchy's integral formula.

Theorem 7. *Suppose that f is holomorphic in an open disk D, and let γ to be a closed curve in D. Then for any z_0 not on γ we have*

$$\frac{1}{2\pi \boldsymbol{i}}\int_\gamma \frac{f(z)}{z-z_0}dz = f(z_0)n(\gamma, z_0).$$

Proof. Fix z_0 and consider function

$$F(z) := \frac{f(z)-f(z_0)}{z-z_0}.$$

$F(z)$ is holomorphic in $D-\{z_0\}$ by usual differentiation rules, and since $f(z)$ is continuous at z_0, $F(z)$ also satisfies

$$\lim_{z\to z_0} F(z)(z-z_0) = 0.$$

Therefore we can apply Theorem 6 to $F(z)$ in $D-\{z_0\}$ and conclude that

$$\int_\gamma F(z) = \int_\gamma \frac{f(z)-f(z_0)}{z-z_0}dz = 0.$$

Equivalently we can rewrite this identity as

$$\frac{1}{2\pi \boldsymbol{i}}\int_\gamma \frac{f(z)}{z-z_0}dz = f(z_0)\frac{1}{2\pi \boldsymbol{i}}\int_\gamma \frac{dz}{z-z_0}.$$

But the latter term is exactly $f(z_0)n(\gamma, z_0)$. □

In a special case $n(\gamma, z_0) = 1$ we have

$$f(z_0) = \frac{1}{2\pi \boldsymbol{i}}\int_\gamma \frac{f(z)}{z-z_0}dz,$$

or after renaming variables in a more common way:

$$f(z) = \frac{1}{2\pi \boldsymbol{i}}\int_\gamma \frac{f(\zeta)}{\zeta-z}d\zeta.$$

It is this formula which is usually referred to as *Cauchy's integral formula*. We must remember that it is valid only when $n(\gamma, z) = 1$ and that we have proved it only when $f(z)$ is analytic in a disk D.

The key point of this theorem is that we can recover completely function $f(z)$ only knowing it on the curve γ.

Example 8. Let γ be the unit circle oriented clockwise, then we have

$$\int_\gamma \frac{e^z}{z}dz = 2\pi\boldsymbol{i}\cdot e^0 = 2\pi\boldsymbol{i}.$$

Example 9. Consider the integral $\int_{|z|=2}\frac{2dz}{1+z^2}$. Decomposing the function under the integral into partial fractions we find:

$$\int_{|z|=2}\frac{2dz}{1+z^2} = \int_{|z|=2}\left(\frac{\boldsymbol{i}}{z+\boldsymbol{i}} - \frac{\boldsymbol{i}}{z-\boldsymbol{i}}\right)dz = \boldsymbol{i}\cdot 2\pi\boldsymbol{i} - \boldsymbol{i}\cdot 2\pi\boldsymbol{i} = 0.$$