Lecture 7

Applications of Cauchy's Integral Formula

Last time we have proved that for a function $f(z)$ holomorphic in an open disk D, and a closed curve $\gamma \subset D$, for any $z \in D$ such that $n(\gamma, z) = 1$ we have

$$f(z) = \frac{1}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1}$$

For concreteness, we assume that $D = B_R(z_0)$ and $\gamma = \partial B_R(z_0) = \{|z - z_0| = r,\ r < R\}$ so that the above formula holds for any $z \in D_r(z_0)$, since in this case $n(\gamma, z) = 1$.

Today we will use this Cauchy's formula to derive important consequences about the structure of holomorphic functions.

Higher derivatives

Provided that we can recursively differentiate identity (1) under the integral sign we would find:

$$f^{(n)}(z) = \frac{n!}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n \in \mathbb{N}. \tag{2}$$

Once we can justify differentiation under the integral sign, we will prove the following key theorem.

Theorem 1. *If function $f(z)$ is holomorphic in an open region $U \subset \mathbb{C}$ then $f(z)$ has complex derivatives of all orders. Moreover for any $z \in U$ and $B_r(z_0)$ such that*

$$z \in B_r(z_0) \subset \overline{B_r(z_0)} \subset U$$

we have

$$f^{(n)}(z) = \frac{n!}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n \in \mathbb{N},$$

where $\gamma = \{|z - z_0| = r\}$ is the boundary of $B_r(z_0)$.

Proof. For any point $z_0 \in U$ there exists an open disk D which contains z_0 and is contained in U with its closure. Then we can apply (2) and arrive at the conclusion of the theorem. □

The following lemma justifies differentiation under the integral sign.

Lemma 2. *Suppose that $\varphi(\zeta)$ is continuous on the curve γ. Then the functions*

$$F_n(z) := \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta$$

are holomorphic in each of the regions cut out by γ and satisfy

$$F_n'(z) = nF_{n+1}'(z).$$

Proof. Fix $z \in \mathbb{C} - \gamma$ and take $r > 0$ such that $B_{2r}(z)$ does not intersect γ. Choose $z + h$ in $B_r(z)$ so that for any ζ on γ we have

$$|(\zeta - z)^{-1}| < r^{-1}, \quad |(\zeta - z - h)^{-1}| < r^{-1}$$

and

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - z - h} \right| = \left| \frac{h}{(\zeta - z)(\zeta - z - h)} \right| < |h| r^{-2}. \tag{3}$$

Consider expression

$$F_n(z + h) - F_n(z) = \int_\gamma \left[\frac{1}{(\zeta - z - h)^n} - \frac{1}{(\zeta - z)^n} \right] \varphi(\zeta) d\zeta.$$

Using identity $A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$ applied to $A = (\zeta - z - h)^{-1}$ and $B = (\zeta - z)^{-1}$, we can rewrite the expression in the brackets as

$$G(\zeta - z, h) := \frac{h}{(\zeta - z - h)(\zeta - z)} \sum_{k=1}^{n} \frac{1}{(\zeta - z - h)^{n-k}(\zeta - z)^{k-1}} = h \sum_{k=1}^{n} \frac{1}{(\zeta - z - h)^{n-k+1}(\zeta - z)^{k}}$$

Inequality (3) implies that $(\zeta - z - h)^{-1} \to (\zeta - z)^{-1}$ as $h \to 0$ uniformly in $(\zeta - z)$. Therefore, given any $\epsilon > 0$ we can choose h small enough so that

$$\left| \frac{1}{h} G(\zeta - z, h) - \frac{n}{(\zeta - z)^{n+1}} \right| < \epsilon$$

everywhere on γ.

Hence

$$\left| \frac{F_n(z+h) - F_n(z)}{h} - n \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}} d\zeta \right| = \left| \int_\gamma \left(\frac{1}{h} G(\zeta, z, h) - \frac{n}{(\zeta - z)^{n+1}} \right) \varphi(\zeta) d\zeta \right| \leqslant \epsilon \cdot \mathrm{Length}(\gamma) \cdot \max(|\varphi(\zeta)|)$$

Letting $\epsilon \to 0$ we conclude that $F_n'(z)$ exists and equals

$$F_n'(z) = n \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}} d\zeta = nF_{n+1}(z).$$

□

With the use of Theorem 1 we can prove the following characterization of holomorphic functions, which gives the converse of Cauchy's theorem.

Theorem 3 (Morera's theorem). *If $f(z)$ is continuous in open $U \subset \mathbb{C}$ and satisfies*

$$\int_\gamma f(z) dz = 0$$

for any closed loop $\gamma \subset U$, then $f(z)$ is holomorphic.

Proof. By fundamental theorem of calculus, the assumption of the theorem implies that $f(z)$ has a primitive $F(z)$. Function $F(z)$ is holomorphic, therefore by Theorem 1 it has complex derivatives of all orders. In particular, $F'(z) = f(z)$ also has a complex derivative. □

Corollary 4. *If a sequence of holomorphic functions $f_n \colon U \to \mathbb{C}$ converge to a function f uniformly on compact subsets $K \Subset U$, then the limiting function $f(z)$ is holomorphic.*

Proof. Being holomorphic is a local property, so we can assume that f_n converge to f uniformly on a closed disk $\overline{D}$. Then for any loop $\gamma \subset D$ we have

$$\int_\gamma f_n(z) dz \to \int_\gamma f(z) dz.$$

However, all the integrals in the sequence vanish since $f_n(z)$ are holomorphic. Therefore the limit integral is also zero. Since loop $\gamma \subset D$ is arbitrary, function $f(z)$ is holomoprhic in D by Morera's theorem. □

This corollary gives a powerful tool for constructing holomorphic functions. For example, it immediately implies that the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

provides a well defined holomorphic function in the region $\mathfrak{Re}(s) > 1$.

Removable singularities

Cauchy's integral formula could be used to extend the domain of a holomorphic function.

Theorem 5. *Let $f(z)$ be holomorphic in $U - \{a\}$. Then $f(z)$ extends to a holomorphic function on the whole U if an only if*

$$\lim_{z\to a}(z-a)f(z) = 0.$$

Proof. Necessity of this assumption is clear, since $f(z)$ has to be continuous at a.

To prove sufficiency, let us enclose a in a small circle $\gamma \subset U$. Then we still have Cauchy's integral formula

$$f(z) = \frac{1}{2\pi \boldsymbol{i}}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

for any $z \neq a$. On the other hand, the right-hand side is an analytic function of z which is well-defined everywhere in the interior of γ, including $z = a$. Consequently, if we define

$$f(a) := \frac{1}{2\pi \boldsymbol{i}}\int_\gamma \frac{f(\zeta)}{\zeta - a}d\zeta,$$

we will obtain a holomorphic function in the entire region U. □

Example 6. If $f(z)$ is holomorphic in U and $f(z_0) = 0$, then $F(z) = \frac{f(z)}{z-z_0}$ extends to a holomorphic function in U with $F(z_0) = \lim_{z\to z_0} F(z) = f'(z_0)$.

The largest $n \in \mathbb{N}$ such that $f(z)/(z-z_0)^n$ extends to a holomorphic function at z_0 is called *the order* of a zero of $f(z)$ at $z = z_0$.

If a holomorphic function $f(z)$ has a zero of order n at z_0, then we can factor $f(z)$ as $f(z) = (z-z_0)^n g(z)$, where $g(z)$ is holomorphic and $g(z_0) \neq 0$.

Cauchy's estimates

An important consequence of Cauchy's formula for higher derivatives of a holomorphic functions is that we can effectively bound $f^{(n)}(z)$ in the interior of a disk D, knowing only a bound on $f(z)$ on the boundary ∂D.

Theorem 7. *Let $\gamma = \partial B_R(z_0)$. Assume that $f(z)$ is holomorphic in a neighbourhood of $B_R(z_0)$, and that $|f(z)| \leqslant M$ on γ. Then*

$$|f^{(n)}(z_0)| \leqslant M n! R^{-n}.$$

Proof. By Cauchy's formula, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi \boldsymbol{i}}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}d\zeta.$$

Under assumptions of the theorem, we have: $|f(\zeta)| \leqslant M$ on γ, $|(\zeta - z_0)^{n+1}| = R^{n+1}$, and $\text{Length}(\gamma) = 2\pi R$. Combining these inequalities we get the conclusion. □

Remark 8. To squeeze the best estimate from the above theorem it is often important to choose R wisely, so that $(\max_{|z-z_0|=R} f(z)) \cdot R^{-n}$ is as small as possible.

The following classical result is an easy consequence of Cauchy estimate for $n = 1$.

Theorem 9 (Liouville's theorem)**.** *If function $f(z)$ is holomorphic and bounded in the entire $\mathbb{C}$, then $f(z)$ is a constant.*

Proof. Assume that $|f(z)| \leqslant M$ for any $z \in \mathbb{C}$. By Cauchy's estimate for $n = 1$ applied to a circle of radius R centered at z, we have

$$|f'(z)| \leqslant M n! R^{-1}.$$

Since z and R are arbitrary, we have $f'(z) = 0$ everywhere in $\mathbb{C}$. Therefore $f(z)$ is a constant. □

Liouille's theorem gives probably the shortest proof of the Fundamental Theorem of Algebra.

Theorem 10 (Fundamental Theorem of Algebra)**.** *Let $P(z)$ be a polynomial of degree $\deg P \geqslant 1$. Then $P(z)$ has a root in $\mathbb{C}$.*

Proof. Assume on the contrary that $P(z)$ never vanish. Then $1/P(z)$ is a holomorphic function in the entire $\mathbb{C}$. Moreover, since $P(z) \to \infty$ as $z \to \infty$, we have that $1/P(z)$ must be bounded. By Liouville's theorem $1/P(z)$ is a constant which contradicts the assumption $\deg P \geqslant 1$. □