Lecture 8

Last time we have derived several important consequences of Cauchy's integral formula. In particular, we have proved that a holomorphic function has infinitely many derivatives given by a formula:

$$f^{(n)}(z) = \frac{n!}{2\pi \boldsymbol{i}} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad n \in \mathbb{N}.$$

Next, we used this formula to derive Cauchy's estimates:

$$|f^{(n)}(a)| \leqslant M n! R^{-n}$$

for a function holomorphic in $B_R(a)$ with $|f(z)| < M$.

As a consequence, any bounded holomorphic function must be a constant (Liouville's theorem). This fact gives a simple proof of the Fundamental Theorem of Algebra.

Taylor's series

Today we will further use Cauchy's integral formula to prove that any function $f(z)$ holomorphic in an open disk $B_R(a)$ can be represented by a convergent power series in that disk. In other words, $f(z)$ is *analytic* in $B_R(a)$. As we know from previous lectures, any power series is holomorphic inside its disk of convergence. For this reason, *analytic* is often used as a synonym for *holomorphic*.

Assume that $f(z)$ is holomorphic in $B_R(a)$ and choose any $r < R$. Function $f(z)$ is continuous in the closed disk $\overline{B_R(a)}$, so we can define

$$M_r := \max_{|z-a| \leqslant r} |f(z)|.$$

Therefore, by Cauchy's estimate, we can conclude that

$$|f^{(n)}(a)| \leqslant M_r n! r^{-n}.$$

Now let us form a power series

$$S(z) := \sum_{n=0}^{\infty} a_n (z-a)^n, \quad a_n = \frac{f^{(n)}(a)}{n!} \tag{1}$$

It follows from Cauchy's estimates that

$$\limsup_{n\to\infty} a_n^{1/n} \leqslant 1/r,$$

therefore series (1) is absolutely convergent in $B_r(a)$. Since $r < R$ is arbitrary, this series is also convergent in $B_R(a)$. Hence, to any function $f(z)$ holomorphic in $B_R(a)$ we can associate a power series (1) convergent in $B_R(a)$.

Question. *Does series $S(z)$ represent function $f(z)$ in the disk $B_R(a)$?*

Today we will demonstrate that the answer is yes, also proving a refined version of the identity $f(z) = S(z)$.

Taylor's theorem

Given any function $f(z)$ holomorphic in an open region U, we can form a new holomorphic function in a punctured region:

$$f_1(z) := \frac{f(z) - f(a)}{z - a}, \quad z \neq a.$$

This function has a *removable singularity* at $z = a$. Therefore, by continuity, we can set $f_1(a) := \lim_{z\to a} f_1(z) = f'(a)$ and get a function holomorphic in the whole U.

We can repeat this process and define a sequence of holomorphic functions

$$f_{k+1}(z) := \frac{f_k(z) - f_k(a)}{z - a}, \quad f_{k+1}(a) = f_k'(a).$$

Unwinding definitions of $f_k(z)$, we arrive at a formula for $f(z)$:

$$f(z) = f(a) + f_1(a)(z-a) + f_2(a)(z-a)^2 + \cdots + f_{n-1}(a)(z-a)^{n-1} + f_n(z)(z-a)^n. \tag{2}$$

Remark 1. Note that in every term except for the last one we have f_k evaluated at a, while the last term f_n is evaluated at z.

Differentiating equation (2) k times $k = 1\dots,n$, and substituting $z = a$, we find $f_k(a) = k!f^{(k)}(a)$, and arrive at the following

Theorem 2 (Taylor's theorem)**.** *Given function $f(z)$ holomorphic in an open set U, point $a \in U$ and $n \in \mathbb{N}$, there exists a holomorphic function $f_n(z)$ such that*

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n. \tag{3}$$

This theorem gives a finite development of any holomorphic function. One of the advantages of this formula is that it makes sense everywhere in the whole domain of holomorphicity of U. In most cases this identity turns out to be more useful and precise then the Taylor's infinite series due to an explicit expression for $f_n(z)$.

Let C be a circle enclosing z and a.

Proposition 3. *Function $f_n(z)$ of (3) is given by a contour integral*

$$f_n(z) = \frac{1}{2\pi \boldsymbol{i}} \int_C \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)} d\zeta.$$

Proof. Cauchy's integral formula applied to $f_n(z)$ states that

$$f_n(z) = \frac{1}{2\pi \boldsymbol{i}} \int_C \frac{f_n(\zeta)}{\zeta - z} d\zeta.$$

Now, let us use (3) to express $f_n(\zeta)$ as

$$f_n(\zeta) = \frac{f(\zeta)}{(\zeta-a)^n} - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!(\zeta-a)^{n-k}}$$

Substituting $f_n(\zeta)$ back into Cauchy's formula, we will have one integral containing $f(\zeta)$, which yields the expected term

$$\frac{1}{2\pi \boldsymbol{i}} \int_C \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)} d\zeta.$$

The remaining terms up to a constant multiple are all of the form:

$$I_r(a) := \int_C \frac{1}{(\zeta-z)(\zeta-a)^r} d\zeta$$

Exercise 1. Prove that all $I_r(a)$ vanish as long as both z and a are inside of the circle C.

Hint: For $r = 1$ this is essentially the statement of a homework assignment. For $r > 1$ we have $I_r'(a) = rI_{r+1}(a)$

This exercise finishes the proof. □

Proposition 3 allows us to get a good control of the reminder term in the finite Taylor's formula.

Taylor's series

Assume that $f(z)$ is holomorphic in an open disk $B_R(a)$. Fix $R' < R$. Let us again start with the Cauchy's integral formula representing $f(z)$ in $B_{R'}(a)$:

$$f(z) = \frac{1}{2\pi \boldsymbol{i}} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where C is a circle of radius R' centered at a. Fix $r < R'$ and assume that $|z-a| < r$.

We can rewrite the factor under the integral as follows

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-a)-(z-a)} = \frac{1}{\zeta-a} \cdot \frac{1}{1-\frac{z-a}{\zeta-a}} = \frac{1}{\zeta-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n,$$

where we used the fact that as long as $|z-a| < r$, we have

$$\left|\frac{z-a}{\zeta-a}\right| < \frac{r}{R'} < 1,$$

and the infinite series converges absolutely and uniformly in $B_r(a)$. Therefore we can plug this series into the Cauchy's formula and interchange summation and integration:

$$\begin{aligned} f(z) &= \frac{1}{2\pi \boldsymbol{i}}\int_C \frac{f(\zeta)}{\zeta-z}d\zeta \\ &= \frac{1}{2\pi \boldsymbol{i}}\int_C \left\{\frac{f(\zeta)}{\zeta-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n\right\}d\zeta \\ &= \frac{1}{2\pi \boldsymbol{i}}\sum_{n=0}^{\infty}(z-a)^n\int_C \frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta \\ &= \sum_{n=0}^{\infty}(z-a)^n\frac{f^{(n)}(a)}{n!}, \end{aligned}$$

where in the last identity we used Cauchy's integral formula for higher derivatives. Therefore everywhere in $B_r(a)$ function $f(z)$ is given by an absolutely convergent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n, \quad a_n = \frac{f^{(n)}(a)}{n!}.$$

Since $r < R' < R$ are arbitrary, the same formula holds everywhere in $B_R(a)$.

Theorem 4. *If function $f(z)$ is holomorphic in $B_R(a)$, then everywhere in $B_R(a)$ function $f(z)$ is represented by its Taylor series and convergence is absolute and uniform in every $B_r(a) \subsetneq B_R(a)$.*

Corollary 5. *If the power series representing $f(z)$ around $z = a$ has radius of convergence R, then $f(z)$ can not be extended to a holomorphic function in a neighbourhood of $\overline{B_R(a)}$.*

Rigidity of holomorphic functions

Lemma 6. *If a holomorphic function $f(z)$ has $f^{(k)}(a) = 0$ for $k = 0, \dots n-1$, then there exists a factorization*

$$f(z) = (z-a)^n g(z)$$

where $g(z)$ is a holomorphic function.

Proof. This lemma immediately follows from the finite Taylor's formula. □

Definition 7. The maximal n such that there exists a factorization

$$f(z) = (z-a)^n g(z)$$

with holomorphic $g(z)$, is called the *order* of the zero at a. If a is a zero of order n then $g(a) \neq 0$.

Theorem 8. *If function $f(z)$ is holomorphic in an open connected region $U \subset \mathbb{C}$ and there exists a sequence of points $\{a_n\}$ converging to $a \in U$, such that $a_n \neq a$ and $f(a_n) = 0$, then $f(z)$ is identically zero in U.*

Proof. Step 1. First we prove that all derivatives of $f(z)$ at $z = a$ are zero. Indeed let $n \in \mathbb{N}$ be the largest number such that $f^{(k)}(a) = 0$ for all $k < n$. If $n = \infty$, then we are done. Otherwise, by the previous lemma we can write $f(z)$ as

$$f(z) = (z-a)^n g(z).$$

Since $f(a_n) = 0$, and $a_n \neq a$, we have $g(a_n) = 0$. Moreover, as $a_n \to a$, and $g(z)$ is continuous, $g(a) = 0$. Therefore, we can further factor $g(z)$ as $g(z) = (z-a)h(z)$. This contradicts maximality of n.

Step 2. In a disk around point $z = a$ function $f(z)$ is represented by an identically zero power series, therefore $f(z)$ is also identically zero in that disk.

Step 3. It remains to prove that $f(z)$ is identically zero in the whole U. To prove it, we introduce the set

$$W = \{z_0 \in U \mid \text{all derivatives } f^{(n)}(z_0) \text{ are zero}\}.$$

Clearly $a \in W$. Moreover W is closed by continuity and open, since if $z_0 \in W$, then $f(z)$ is identically zero in a small disk around z_0. Therefore $W = U$. □

Corollary 9. *If holomorphic functions $f(z)$ and $g(z)$ coincide on a subset $Z \subset U$ which has an accumulation point $a \in U$ then $f(z)$ and $g(z)$ coincide everywhere in U.*