Lecture 9

Recall that point $a \in Z \subset \mathbb{C}$ is called an *isolated* point of Z if there exists a small neighbourhood $B_\epsilon(a)$ such that $B_\epsilon(a) \cap Z = \{a\}$. Otherwise, point a is an *accumulation* point of Z, meaning that there exists a sequence of points $a_n \in Z$, $a_n \neq a$ such that $a_n \to a$.

Last time we have proved

Theorem 1. *If holomorphic functions $f(z)$ and $g(z)$ coincide on a subset $Z \subset U$ which has an accumulation point $a \in U$ then $f(z)$ and $g(z)$ coincide everywhere in U.*

Remark 2. Some specific important situations when the above theorem can be applied are: when Z is an open disk, or an arc.

As a consequence of the above theorem, we can conclude that if $f(z)$ is a *nonzero* holomorphic function then its zeros are isolated. Moreover, if z_0 is a zero of such function, then there exists a number $n \geqslant 0$ such that $f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ while $f^{(n)}(z_0) \neq 0$. Equivalently, $f(z)$ can be factored as

$$f(z) = (z - z_0)^n g(z), \quad g(z_0) \neq 0, \tag{1}$$

where $g(z)$ is a holomorphic function.

Singularities of holomorphic functions

Let $f(z)$ be a holomorphic function defined in a neighbourhood U of point z_0, except for point z_0 itself. We will say that $f(z)$ has an *isolated singularity* at z_0. The purpose of today's lecture to provide a characterization of different types of isolated singularities.

Removable singularities

Definition 3. Function $f(z)$ has a *removable singularity* at z_0 if

$$\lim_{z \to z_0} (z - z_0) f(z) = 0. \tag{2}$$

In one of the previous lectures we have proved that if has a removable singularity at z_0, then, we can define $f(z_0)$ in such a way that $f(z)$ is holomorphic in the whole region U. In particular, there must exist a limit

$$\lim_{z \to z_0} f(z)$$

and this limit recovers the value $f(z_0)$.

Example 4. For a holomorphic function $f(z)$ such that $f(z_0) = 0$, a function $F(z) := \frac{f(z)}{z - z_0}$ has a removable singularity at $z = z_0$. The value of $F(z)$ can be computed via the limit:

$$\lim_{z \to z_0} F(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Poles

Definition 5. We will say that function $f(z)$ has a *pole* at z_0 if

$$\lim_{z \to z_0} f(z) = \infty, \tag{3}$$

i.e., for any $R > 0$ there exists $\epsilon > 0$ such that for any $z \in B_\epsilon(z_0) - \{z_0\}$ we have $|f(z)| > R$.

Example 6. Function $f(z) = 1/z$ has a pole at $z = 0$. More generally, any rational function represented by an *irreducible* fraction $\frac{P(z)}{Q(z)}$ has poles at zeros of $Q(z)$.

Remark 7. While conditions (2) and (3) are not mutually exclusive apriori, theorem on removable singularities guarantees that removable singularity can't be a pole.

Now, we get a refined local characterization of poles. Assuming that $f(z)$ has a pole at $z = z_0$, we can find a small neighbourhood U around z_0 such that $f(z) \neq 0$ in U. Then function $h(z) := 1/f(z)$ is holomorphic in $U - \{z_0\}$ and satisfies

$$\lim_{z \to z_0} h(z) = 0.$$

Therefore $h(z)$ has removable singularity at $z = z_0$ with $h(z_0) = 0$. Now, using factorization (1), we can write $h(z)$ as

$$h(z) = (z - z_0)^n g(z),$$

where $g(z_0) \neq 0$. In particular, we can write $g(z) = 1/F(z)$, where $F(z)$ is holomorphic in U and $F(z_0) \neq 0$. Switching back to $f(z)$ we find that

$$f(z) = \frac{F(z)}{(z - z_0)^n},$$

where $F(z_0) \neq 0$. Therefore, we have proved the following:

Theorem 8. *If function $f(z)$ has a pole at z_0, then there exists a holomorphic function $F(z)$ and a number $n \in \mathbb{N}$ such that*

$$f(z) = \frac{F(z)}{(z - z_0)^n}. \tag{4}$$

*Number n is called the **order** of pole z_0.*

Remark 9. To be precise, we have proved that (4) holds in a small neighbourhood of z_0. But we can extend it to the whole U just by setting $F(z) := f(z)(z - z_0)^n$.

Definition 10. Function $f(z)$ is *meromorphic* in U if it is holomorphic in U except for isolated poles.

Example 11. Any rational function is meromorphic in $\mathbb{C}$.

If function $f(z)$ has a pole of order n at z_0, we can use Taylor's formula to expand $F(z) := (z - z_0)^n f(z)$:

$$F(z) = a_n + a_{n-1}(z - z_0) + \cdots + a_1(z - z_0)^{n-1} + \varphi(z)(z - z_0)^n,$$

where $\varphi(z)$ is holomorphic in U. Hence, in $U - \{z_0\}$ we have

$$f(z) = \boxed{\frac{a_N}{(z - z_0)^N} + \cdots + \frac{a_1}{(z - z_0)}} + \varphi(z) \tag{5}$$

where the *boxed* terms form *the singular* or *principle* part of $f(z)$ at $z = z_0$. Representation (5) allows to work with poles via their principle parts as if we were working with ordinary rational functions.

Example 12. Function $f(z) = \dfrac{1}{1 - e^{-z}}$ has a *simple* (i.e., of order 1) pole at $z = 0$. Indeed, by Taylor's formula

$$e^{-z} = 1 - z\varphi(z),$$

where $\varphi(z)$ is holomorphic, $\varphi(0) \neq 0$, therefore we have a representation of $f(z)$

$$f(z) = \frac{1/\varphi(z)}{z}$$

with holomorphic numerator.

Essential singularity

Definition 13. If an isolated singularity z_0 is neither a removable singularity, nor a pole, then we say that z_0 is an *essential singularity*.

In other words, if there is no finite or infinite limit

$$\lim_{z \to z_0} f(z),$$

then z_0 is an essential singularity.

Example 14. Function $f(z) = e^{1/z}$ has an essential singularity at $z = 0$. Indeed, as z tends to zero along positive real ray, $f(z)$ approaches ∞, while as z tends to zero along negative real ray, $f(z)$ approaches 0.

Any holomorphic function at an essential singularity has an extremely wild behavior.

Theorem 15 (Great Picard's theorem). *If a holomorphic function $f(z)$ has an essential singularity at a point z_0, then on any punctured neighborhood of z_0, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.*

This is a difficult theorem, and we do not have necessary techniques to prove it. Instead, we will prove a much weaker statement of a similar spirit:

Theorem 16 (Casorati-Weierstrass). *If a holomorphic function $f(z)$ has an essential singularity at a point z_0, then the image of any punctured neighborhood of z_0 is everywhere dense in $\mathbb{C}$.*

Proof. Assume the contrary: that is there exists $\zeta \in \mathbb{C}$ and $\epsilon > 0$ such that $f(U - \{z_0\})$ does not intersect $B_\epsilon(\zeta)$.

Let us introduce a new function $g(z) = \frac{1}{f(z)-\zeta}$. Since the values of $f(z)$ in $U - \{z_0\}$ are separated from ζ, we will have that $g(z)$ is bounded in $U - \{z_0\}$. Therefore z_0 is a removable singularity of $g(z)$ and we can define $g(z_0)$ which makes $g(z)$ holomorphic in U.

Then $f(z) = \frac{1}{g(z)} + \zeta$ has either a removable singularity (if $g(z_0) \neq 0$) or a pole (if $g(z_0) = 0$) at z_0 which contradicts our assumption that z_0 is an essential singularity. □

Isolated singularity at ∞

All of the above discussion makes sense for a function $f(z)$ holomorphic in a neighbourhood of $\infty \in \hat{\mathbb{C}}$ on the extended complex plane[1]. In this case, we can consider a function $F(z) := f(1/z)$ which would have an isolated singularity at $z = 0$.

Definition 17. We will say that a function $f(z)$ has a removable/pole/essential singularity at ∞ if $F(z) := f(1/z)$ has removable/pole/essential singularity at $z = 0$.

Example 18. A polynomial $P(z)$ of positive degree has a pole at ∞ of order $\deg P$.

Rational function $P(z)/Q(z)$ has a removable singularity at ∞ if and only if $\deg P \leqslant \deg Q$.

Theorem 19. *The meromorphic functions in the extended complex plane $\hat{\mathbb{C}}$ are rational functions.*

Proof. First, we claim that a meromorphic function in $\hat{\mathbb{C}}$ has a finite number of poles. Indeed, by definition, poles are isolated, and any discrete subset of $\hat{\mathbb{C}}$ must be finite, since $\hat{\mathbb{C}}$ is compact.

Once we know that the number of poles is finite, we can essentially repeat the proof of the theorem about partial fraction expansion of a rational function.

Let $\{z_1, \dots z_N\}$ be the set of poles of $f(z)$. Denote by $p_i(z)$ principle part of a pole z_i. Clearly $p_i(z)$ is a rational function. Therefore, we can write $f(z)$ as

$$f(z) = Q(z) + g(z).$$

where $Q(z) = \sum p_i(z)$ is a rational function, and $g(z)$ is a holomorphic function (meromorphic function without poles) defined in the entire extended complex plane $\hat{\mathbb{C}}$. Since $\hat{\mathbb{C}}$ is compact, $g(z)$ must be bounded, so Liouville's theorem implies that $g(z)$ is constant. □

[1] Recall that a neighbourhood of ∞ is a complement of a compact set $K \subset \mathbb{C} \subset \mathbb{C} \cup \{\infty\}$.