Midterm

Your final score for the midterm is $\min(50, \sum_{i=1}^{6} s_i)$ where $s_i \in [0;10]$ is your score for the i-th problem. To get the full credit, write complete, detailed solutions. You may use any of the results from the class without a proof, but you have to state them explicitly.

Problem 1. Which of the following are holomorphic functions of $z = x + \boldsymbol{i}y$

a) $f(z) = x^2 + \boldsymbol{i}y^2$;

b) $f(z) = x^2 - y^2 + \boldsymbol{i}2xy$;

c) $f(z) = e^y(\cos x + \boldsymbol{i}\sin x)$?

Problem 2. Describe the image of the complex half-plane $\{\mathfrak{Re}(z) > 0\}$ under the map $f(z) = \sqrt{z^2+1}$, where $\sqrt{w}$ is the *principle branch* of the square root of $w \in \mathbb{C} - (-\infty; 0]$

Problem 3. Function $f(z)$ is holomorphic in $\mathbb{C} - \{0\}$, has a pole of order 1 at $z = 0$, and there exists $R > 0$ such that

$$|f(z)| < |z|^{3/2}$$

as long as $|z| > R$. Classify all such functions $f(z)$.

Problem 4. For a continuous function $\varphi\colon \overline{\mathbb{D}} \to \mathbb{C}$ let us introduce a "non-holomorphicity measure"

$$m(\varphi) = \inf_f \sup_{z\in\overline{\mathbb{D}}} |\varphi(z) - f(z)|,$$

where the infimum is taken over all functions f holomorphic in a neighbourhood of $\mathbb{D}$. Compute $m(\varphi)$ for $\varphi(z) = |z|$.

Problem 5. Find the number of zeros of the polynomial $q(z) = z^6 - 2z^4 + 6z^3 + z + 1$ inside the unit disk $\mathbb{D}$.

Problem 6. Let $f(z)\colon \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function. Assume that $f(z)$ has finitely many zeros in $\mathbb{C}$. Prove that there exist a polynomial $P(z)$ and an entire holomorphic function $g(z)$ such that

$$f(z) = P(z)e^{g(z)}.$$