Midterm

Problem 1. Which of the following are holomorphic functions of $z = x + \boldsymbol{i}y$

a) $f(z) = x^2 + \boldsymbol{i}y^2$;

b) $f(z) = x^2 - y^2 + \boldsymbol{i}2xy$;

c) $f(z) = e^y(\cos x + \boldsymbol{i}\sin x)$?

Answer: Only b) is holomorphic in $\mathbb{C}$.

Solution: a) and c) do not satisfy Cauchy-Riemann equations in $\mathbb{C}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad u = \mathfrak{Re}(f), v = \mathfrak{Im}(f).$$

Indeed, for a) we have

$$\frac{\partial u}{\partial x} = 2x \neq 2y = \frac{\partial v}{\partial y}.$$

Similarly, for c) we have

$$\frac{\partial u}{\partial x} = -e^y \sin x \neq e^y \sin x = \frac{\partial v}{\partial y}.$$

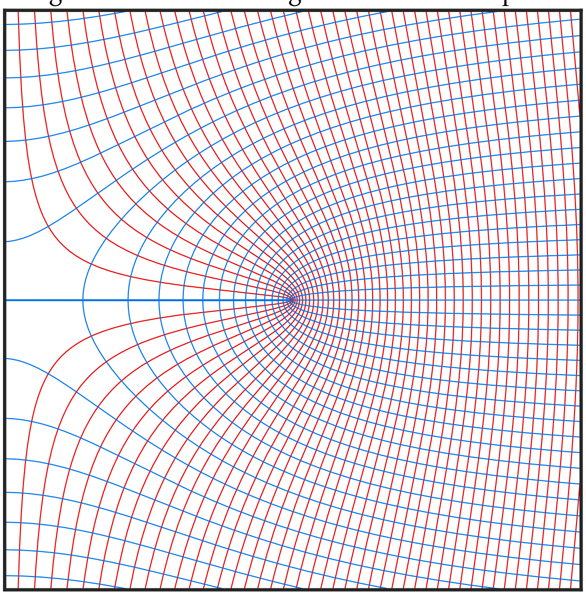
On the other hand, function in b) is $f(z) = z^2$ which is obviously holomorphic. □

Problem 2. Describe the image of the complex half-plane $\{\mathfrak{Re}(z) > 0\}$ under the map $f(z) = \sqrt{z^2+1}$, where $\sqrt{w}$ is the *principle branch* of the square root of $w \in \mathbb{C} - (-\infty; 0]$

Answer: Open right half plane without the unit segment between 0 and 1:

$$\{\mathfrak{Re}(z) > 0\} - (0; 1].$$

Figure 1: Image of the coordinate grid under the map $z \mapsto \sqrt{1+z^2}$.



Solution: We treat f as the composition of 3 mappings:

$$z \mapsto z^2 \mapsto z^2 + 1 \mapsto \sqrt{z^2+1}.$$

The first map transforms the right half plane bijectively onto the complement of the negative x-axis. Indeed

$$\{\mathfrak{Re}(z) > 0\} = \{z \mid \operatorname{Arg}(z) \in (-\pi/2, \pi/2)\}.$$

Since $z \mapsto z^2$ doubles the argument, the image of $\{\mathfrak{Re}(z) > 0\}$ is $\{z \mid \mathrm{Arg}(z) \in (-\pi, \pi)\} = \mathbb{C} - (-\infty; 0]$

Map $w \mapsto w + 1$ transforms $\mathbb{C} - (-\infty; 0]$ into $\mathbb{C} - (-\infty; 1]$.

Finally, since $\sqrt{w}$ transforms $\mathbb{C} - (-\infty; 0]$ bijectively onto the right half plane, the image of $\mathbb{C} - (-\infty; 1]$ under $w \mapsto \sqrt{w}$ ca be found as the image of $\mathbb{C} - (-\infty; 0]$ minus the image of $(0; 1]$ which is again $(0; 1]$:

$$f(\{\mathfrak{Re}(z) > 0\}) = \{\mathfrak{Re}(z) > 0\} - (0; 1].$$

□

Problem 3. Function $f(z)$ is holomorphic in $\mathbb{C} - \{0\}$, has a pole of order 1 at $z = 0$, and there exists $R > 0$ such that

$$|f(z)| < |z|^{3/2}$$

as long as $|z| > R$. Classify all such functions $f(z)$.

Answer: $f(z) = az + b + \frac{c}{z}$, where $a, b, c \in \mathbb{C}$ with $c \neq 0$.

Solution: Since $f(z)$ has simple pole at $z = 0$, we can factor $f(z)$ as

$$f(z) = \frac{g(z)}{z}$$

where $g(z)$ is an entire holomorphic function with $g(0) \neq 0$.

We know that $|g(z)| < |z|^{5/2}$ for all large $|z|$. By Cauchy's estimate for higher derivatives, we conclude

$$|g^{(k)}(0)| \leqslant Mk!R^{-k} \leqslant k!R^{5/2-k}, \quad M = \sup_{|z|=R} |f(z)|.$$

Considering $R \to +\infty$, we conclude that for all $k \geqslant 3$ we have $g^{(k)}(0) = 0$. So $g(z)$ is a quadratic polynomial: $g(z) = az^2 + bz + c$ and

$$f(z) = \frac{g(z)}{z} = az + b + \frac{c}{z}.$$

Since $g(0) \neq 0$ we have $c \neq 0$, and clearly any function of the above form satisfies all the properties of the problem. □

Problem 4. For a continuous function $\varphi \colon \overline{\mathbb{D}} \to \mathbb{C}$ let us introduce a "non-holomorphicity measure"

$$m(\varphi) = \inf_f \sup_{z \in \overline{\mathbb{D}}} |\varphi(z) - f(z)|,$$

where the infimum is taken over all functions f holomorphic in a neighbourhood of $\mathbb{D}$. Compute $m(\varphi)$ for $\varphi(z) = |z|$.

Answer: $m(|z|) = 1/2$.

Solution:

First, note that for $f(z) \equiv 1/2$ we have

$$\sup_{z \in \overline{\mathbb{D}}} ||z| - f(z)| = 1/2,$$

so $m(|z|) \leqslant 1/2$.

Now, for any holomorphic function $f(z)$ we have

$$\sup_{z \in \overline{\mathbb{D}}} ||z| - f(z)| \geqslant \sup_{|z|=1} |1 - f(z)| \qquad \text{and} \qquad \sup_{z \in \overline{\mathbb{D}}} ||z| - f(z)| \geqslant |f(0)|$$

By the maximum modulus principle applied to $1 - f(z)$ we have

$$\sup_{|z|=1} |1 - f(z)| \geqslant |1 - f(0)|$$

Combining the above inequalities, we find:

$$\sup_{z \in \overline{\mathbb{D}}} ||z| - f(z)| \geqslant |1 - f(0)| \geqslant 1 - |f(0)| \geqslant 1 - \sup_{z \in \overline{\mathbb{D}}} ||z| - f(z)|$$

so $\sup_{z \in \overline{\mathbb{D}}} ||z| - f(z)| \geqslant 1/2$ and $m(|z|) = 1/2$. □

Problem 5. Find the number of zeros of the polynomial $q(z) = z^6 - 2z^4 + 6z^3 + z + 1$ inside the unit disk $\mathbb{D}$.

Answer: 3 zeros.

Solution: Summand $f(z) = 6z^3$ dominates $g(z) = z^6 - 2z^4 + z + 1$ on $\{|z| = 1\}$, i.e, $|f(z)| > |g(z)|$ on the unit circle. Hence we can apply Rouché's theorem and conclude that polynomial $q(z) = f(z) + g(z)$ has the same number of zeros in the unit disk as $f(z) = 6z^3$ and the latter has one root with multiplicity 3.

Problem 6. Let $f(z)\colon \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function. Assume that $f(z)$ has finitely many zeros in $\mathbb{C}$. Prove that there exist a polynomial $P(z)$ and an entire holomorphic function $g(z)$ such that

$$f(z) = P(z)e^{g(z)}.$$

Solution: For every zero z_0 of function $f(z)$ we can factor out the term $(z-z_0)^k$ where k is the order of zero at z_0:

$$f(z) = (z - z_0)^k \varphi(z), \quad \varphi(z_0) \neq 0.$$

Identifying such factor for each zero of $f(z)$, we will represent $f(z)$ as

$$f(z) = P(z)h(z),$$

where $P(z)$ is a polynomial in z and $h(z)$ is an entire function with no zeros.

By a general result, a holomorphic function which does not vanish in a simply connected region admits a complex logarithm in that region:

$$h(z) = e^{g(z)},$$

and

$$f(z) = P(z)e^{g(z)}.$$

as required. □